MATHEMATICAL ASSOCIATION OF AMERICA

American Mathematics Competitions

63rd Annual

AMC 12 A

American Mathematics Contest 12 A

Tuesday, February 7, 2012

INSTRUCTIONS

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR TELLS YOU.
2. This is a twenty-five question multiple choice test. Each question is followed by answers marked A, B, C, D and E. Only one of these is correct.
3. Mark your answer to each problem on the AMC 12 Answer Form with a #2 pencil. Check the blackened circles for accuracy and erase errors and stray marks completely. Only answers properly marked on the answer form will be graded.
4. SCORING: You will receive 6 points for each correct answer, 1.5 points for each problem left unanswered, and 0 points for each incorrect answer.
5. No aids are permitted other than scratch paper, graph paper, rulers, compass, protractors, and erasers. No calculators are allowed. No problems on the test will *require* the use of a calculator.
6. Figures are not necessarily drawn to scale.
7. Before beginning the test, your proctor will ask you to record certain information on the answer form.
8. When your proctor gives the signal, begin working on the problems. You will have **75 minutes** to complete the test.
9. When you finish the exam, *sign your name* in the space provided on the Answer Form.

The Committee on the American Mathematics Competitions (CAMC) reserves the right to re-examine students before deciding whether to grant official status to their scores. The CAMC also reserves the right to disqualify all scores from a school if it is determined that the required security procedures were not followed.

Students who score 100 or above or finish in the top 5% on this AMC 12 will be invited to take the 30th annual American Invitational Mathematics Examination (AIME) on Thursday, March 15, 2012 or Wednesday, March 28, 2012. More details about the AIME and other information are on the back page of this test booklet.

2012

AMC 12 A

DO NOT OPEN UNTIL TUESDAY, FEBRUARY 7, 2012

****Administration On An Earlier Date Will Disqualify Your School's Results****

1. All information (Rules and Instructions) needed to administer this exam is contained in the TEACHERS' MANUAL, which is outside of this package. PLEASE READ THE MANUAL BEFORE FEBRUARY 7, 2012. Nothing is needed from inside this package until February 7.
2. Your PRINCIPAL or VICE-PRINCIPAL must verify on the AMC 12 CERTIFICATION FORM (found in the Teachers' Manual) that you followed all rules associated with the conduct of the exam.
3. The Answer Forms must be mailed by trackable mail to the AMC office no later than 24 hours following the exam.
4. *The publication, reproduction or communication of the problems or solutions of this test during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination at any time via copier, telephone, email, internet or media of any type is a violation of the competition rules.*

1. A bug crawls along a number line, starting at -2. It crawls to -6, then turns around and crawls to 5. How many units does the bug crawl altogether?

   (A) 9 **(B)** 11 **(C)** 13 **(D)** 14 **(E)** 15

2. Cagney can frost a cupcake every 20 seconds and Lacey can frost a cupcake every 30 seconds. Working together, how many cupcakes can they frost in 5 minutes?

   (A) 10 **(B)** 15 **(C)** 20 **(D)** 25 **(E)** 30

3. A box 2 centimeters high, 3 centimeters wide, and 5 centimeters long can hold 40 grams of clay. A second box with twice the height, three times the width, and the same length as the first box can hold n grams of clay. What is n?

   (A) 120 **(B)** 160 **(C)** 200 **(D)** 240 **(E)** 280

4. In a bag of marbles, $\frac{3}{5}$ of the marbles are blue and the rest are red. If the number of red marbles is doubled and the number of blue marbles stays the same, what fraction of the marbles will be red?

   (A) $\frac{2}{5}$ **(B)** $\frac{3}{7}$ **(C)** $\frac{4}{7}$ **(D)** $\frac{3}{5}$ **(E)** $\frac{4}{5}$

5. A fruit salad consists of blueberries, raspberries, grapes, and cherries. The fruit salad has a total of 280 pieces of fruit. There are twice as many raspberries as blueberries, three times as many grapes as cherries, and four times as many cherries as raspberries. How many cherries are there in the fruit salad?

   (A) 8 **(B)** 16 **(C)** 25 **(D)** 64 **(E)** 96

6. The sums of three whole numbers taken in pairs are 12, 17, and 19. What is the middle number?

   (A) 4 **(B)** 5 **(C)** 6 **(D)** 7 **(E)** 8

7. Mary divides a circle into 12 sectors. The central angles of these sectors, measured in degrees, are all integers and they form an arithmetic sequence. What is the degree measure of the smallest possible sector angle?

   (A) 5 **(B)** 6 **(C)** 8 **(D)** 10 **(E)** 12

8. An *iterative average* of the numbers 1, 2, 3, 4, and 5 is computed in the following way. Arrange the five numbers in some order. Find the mean of the first two numbers, then find the mean of that with the third number, then the mean of that with the fourth number, and finally the mean of that with the fifth number. What is the difference between the largest and smallest possible values that can be obtained using this procedure?

   (A) $\frac{31}{16}$ **(B)** 2 **(C)** $\frac{17}{8}$ **(D)** 3 **(E)** $\frac{65}{16}$

9. A year is a leap year if and only if the year number is divisible by 400 (such as 2000) or is divisible by 4 but not by 100 (such as 2012). The 200th anniversary of the birth of novelist Charles Dickens was celebrated on February 7, 2012, a Tuesday. On what day of the week was Dickens born?

   (A) Friday **(B)** Saturday **(C)** Sunday **(D)** Monday **(E)** Tuesday

10. A triangle has area 30, one side of length 10, and the median to that side of length 9. Let θ be the acute angle formed by that side and the median. What is $\sin\theta$?

    (A) $\frac{3}{10}$ **(B)** $\frac{1}{3}$ **(C)** $\frac{9}{20}$ **(D)** $\frac{2}{3}$ **(E)** $\frac{9}{10}$

11. Alex, Mel, and Chelsea play a game that has 6 rounds. In each round there is a single winner, and the outcomes of the rounds are independent. For each round the probability that Alex wins is $\frac{1}{2}$, and Mel is twice as likely to win as Chelsea. What is the probability that Alex wins three rounds, Mel wins two rounds, and Chelsea wins one round?

    (A) $\frac{5}{72}$ **(B)** $\frac{5}{36}$ **(C)** $\frac{1}{6}$ **(D)** $\frac{1}{3}$ **(E)** 1

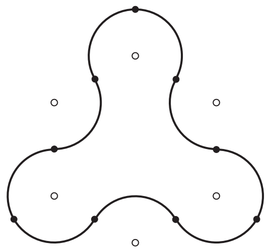
12. A square region $ABCD$ is externally tangent to the circle with equation $x^2+y^2=1$ at the point $(0,1)$ on the side CD. Vertices A and B are on the circle with equation $x^2+y^2=4$. What is the side length of this square?

    (A) $\frac{\sqrt{10}+5}{10}$ **(B)** $\frac{2\sqrt{5}}{5}$ **(C)** $\frac{2\sqrt{2}}{3}$ **(D)** $\frac{2\sqrt{19}-4}{5}$ **(E)** $\frac{9-\sqrt{17}}{5}$

13. Paula the painter and her two helpers each paint at constant, but different, rates. They always start at 8:00 AM and all three always take the same amount of time to eat lunch. On Monday the three of them painted 50% of a house, quitting at 4:00 PM. On Tuesday, when Paula wasn't there, the two helpers painted only 24% of the house and quit at 2:12 PM. On Wednesday Paula worked by herself and finished the house by working until 7:12 PM. How long, in minutes, was each day's lunch break?

(A) 30 **(B)** 36 **(C)** 42 **(D)** 48 **(E)** 60

14. The closed curve in the figure is made up of 9 congruent circular arcs each of length $\frac{2\pi}{3}$, where each of the centers of the corresponding circles is among the vertices of a regular hexagon of side 2. What is the area enclosed by the curve?



(A) $2\pi + 6$ **(B)** $2\pi + 4\sqrt{3}$ **(C)** $3\pi + 4$ **(D)** $2\pi + 3\sqrt{3} + 2$ **(E)** $\pi + 6\sqrt{3}$

15. A 3×3 square is partitioned into 9 unit squares. Each unit square is painted either white or black with each color being equally likely, chosen independently and at random. The square is then rotated 90° clockwise about its center, and every white square in a position formerly occupied by a black square is painted black. The colors of all other squares are left unchanged. What is the probability that the grid is now entirely black?

(A) $\dfrac{49}{512}$ **(B)** $\dfrac{7}{64}$ **(C)** $\dfrac{121}{1024}$ **(D)** $\dfrac{81}{512}$ **(E)** $\dfrac{9}{32}$

16. Circle C_1 has its center O lying on circle C_2. The two circles meet at X and Y. Point Z in the exterior of C_1 lies on circle C_2 and $XZ = 13$, $OZ = 11$, and $YZ = 7$. What is the radius of circle C_1?

(A) 5 **(B)** $\sqrt{26}$ **(C)** $3\sqrt{3}$ **(D)** $2\sqrt{7}$ **(E)** $\sqrt{30}$

17. Let S be a subset of $\{1, 2, 3, \ldots, 30\}$ with the property that no pair of distinct elements in S has a sum divisible by 5. What is the largest possible size of S?

(A) 10 **(B)** 13 **(C)** 15 **(D)** 16 **(E)** 18

18. Triangle ABC has $AB = 27$, $AC = 26$, and $BC = 25$. Let I denote the intersection of the internal angle bisectors of $\triangle ABC$. What is BI?

(A) 15 **(B)** $5 + \sqrt{26} + 3\sqrt{3}$ **(C)** $3\sqrt{26}$ **(D)** $\frac{2}{3}\sqrt{546}$ **(E)** $9\sqrt{3}$

19. Adam, Benin, Chiang, Deshawn, Esther, and Fiona have internet accounts. Some, but not all, of them are internet friends with each other, and none of them has an internet friend outside this group. Each of them has the same number of internet friends. In how many different ways can this happen?

(A) 60 **(B)** 170 **(C)** 290 **(D)** 320 **(E)** 660

20. Consider the polynomial

$$P(x) = \prod_{k=0}^{10} (x^{2^k} + 2^k) = (x + 1)(x^2 + 2)(x^4 + 4) \cdots (x^{1024} + 1024).$$

The coefficient of x^{2012} is equal to 2^a. What is a?

(A) 5 **(B)** 6 **(C)** 7 **(D)** 10 **(E)** 24

21. Let a, b, and c be positive integers with $a \geq b \geq c$ such that

$$a^2 - b^2 - c^2 + ab = 2011 \text{ and}$$
$$a^2 + 3b^2 + 3c^2 - 3ab - 2ac - 2bc = -1997.$$

What is a?

(A) 249 **(B)** 250 **(C)** 251 **(D)** 252 **(E)** 253

22. Distinct planes $p_1, p_2, \ldots, p_k$ intersect the interior of a cube Q. Let S be the union of the faces of Q and let $P = \bigcup_{j=1}^{k} p_j$. The intersection of P and S consists of the union of all segments joining the midpoints of every pair of edges belonging to the same face of Q. What is the difference between the maximum and the minimum possible values of k?

(A) 8 **(B)** 12 **(C)** 20 **(D)** 23 **(E)** 24

23. Let S be the square one of whose diagonals has endpoints $(0.1, 0.7)$ and $(-0.1, -0.7)$. A point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers x and y such that $0 \le x \le 2012$ and $0 \le y \le 2012$. Let $T(v)$ be a translated copy of S centered at v. What is the probability that the square region determined by $T(v)$ contains exactly two points with integer coordinates in its interior?

(A) 0.125 **(B)** 0.14 **(C)** 0.16 **(D)** 0.25 **(E)** 0.32

24. Let $\{a_k\}_{k=1}^{2011}$ be the sequence of real numbers defined by

$$a_1 = 0.201, \quad a_2 = (0.2011)^{a_1}, \quad a_3 = (0.20101)^{a_2}, \quad a_4 = (0.201011)^{a_3},$$

and more generally

$$a_k = \begin{cases} (0.\underbrace{20101\ldots0101}_{k+2 \text{ digits}})^{a_{k-1}}, & \text{if } k \text{ is odd,} \\ (0.\underbrace{20101\ldots01011}_{k+2 \text{ digits}})^{a_{k-1}}, & \text{if } k \text{ is even.} \end{cases}$$

Rearranging the numbers in the sequence $\{a_k\}_{k=1}^{2011}$ in decreasing order produces a new sequence $\{b_k\}_{k=1}^{2011}$. What is the sum of all the integers k, $1 \le k \le 2011$, such that $a_k = b_k$?

(A) 671 **(B)** 1006 **(C)** 1341 **(D)** 2011 **(E)** 2012

25. Let $f(x) = |2\{x\} - 1|$ where $\{x\}$ denotes the fractional part of x. The number n is the smallest positive integer such that the equation

$$nf(xf(x)) = x$$

has at least 2012 real solutions x. What is n?

Note: the fractional part of x is a real number $y = \{x\}$, such that $0 \le y < 1$ and $x - y$ is an integer.

(A) 30 **(B)** 31 **(C)** 32 **(D)** 62 **(E)** 64

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WRITE TO US!

Correspondence about the problems and solutions for this AMC 12 and orders for publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego

2012 AIME

The 30th annual AIME will be held on Thursday, March 15, with the alternate on Wednesday, March 28. It is a 15-question, 3-hour, integer-answer exam. You will be invited to participate only if you score 120 or above or finish in the top 2.5% of the AMC 10, or if you score 100 or above or finish in the top 5% of the AMC 12. Top-scoring students on the AMC 10/12/AIME will be selected to take the 41st Annual USA Mathematical Olympiad (USAMO) on April 24-25, 2012. The best way to prepare for the AIME and USAMO is to study previous exams. Copies may be ordered as indicated below.

PUBLICATIONS

A complete listing of current publications, with ordering instructions, is at our web site: amc.maa.org