MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

63rd Annual

AMC 12 A

American Mathematics Contest 12 A

Tuesday, February 7, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, email, internet or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (E):** The distance from -2 to -6 is $|(-6)-(-2)| = 4$ units. The distance from -6 to 5 is $|5-(-6)| = 11$ units. Altogether the bug crawls $4+11 = 15$ units.

2. **Answer (D):** Because 20 seconds is $\frac{1}{3}$ of a minute, Cagney can frost $5 \div \frac{1}{3} = 15$ cupcakes in five minutes. Because 30 seconds is $\frac{1}{2}$ of a minute, Lacey can frost $5 \div \frac{1}{2} = 10$ cupcakes in five minutes. Altogether they can frost $15 + 10 = 25$ cupcakes in five minutes.

3. **Answer (D):** The volume of the second box is $2 \cdot 3 = 6$ times the volume of the first box. Hence it can hold $6 \cdot 40 = 240$ grams of clay.

4. **Answer (C):** The ratio of blue marbles to red marbles is $3:2$. If the number of red marbles is doubled, the ratio will be $3:4$, and the fraction of marbles that are red will be $\frac{4}{3+4} = \frac{4}{7}$.

5. **Answer (D):**

   For each blueberry in the fruit salad there are 2 raspberries, 8 cherries, and 24 grapes. Thus there are $1+2+8+24 = 35$ pieces of fruit for each blueberry. Because $280 = 35 \cdot 8$, it follows that there are a total of 8 blueberries, $8 \cdot 2 = 16$ raspberries, $8 \cdot 8 = 64$ cherries, and $8 \cdot 24 = 192$ grapes in the fruit salad. Thus there are 64 cherries.

6. **Answer (D):** Let the three whole numbers be $a < b < c$. The set of sums of pairs of these numbers is $(a+b, a+c, b+c) = (12, 17, 19)$. Thus $2(a+b+c) = (a+b)+(a+c)+(b+c) = 12+17+19 = 48$, and $a+b+c = 24$. If follows that $(a, b, c) = (24-19, 24-17, 24-12) = (5, 7, 12)$. Therefore the middle number is 7.

7. **Answer (C):** Let a be the initial term and d the common difference for the arithmetic sequence. Then the sum of the degree measures of the central angles is

   $$a + (a+d) + \cdots + (a+11d) = 12a + 66d = 360,$$

   so $2a + 11d = 60$. Letting $d = 4$ yields the smallest possible positive integer value for a, namely $a = 8$.

8. **Answer (C):** If the numbers are arranged in the order a, b, c, d, e, then the iterative average is

$$\frac{\frac{\frac{\frac{a+b}{2}+c}{2}+d}{2}+e}{2} = \frac{a+b+2c+4d+8e}{16}.$$

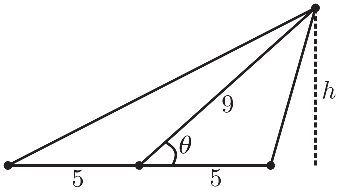
The largest value is obtained by letting $(a,b,c,d,e) = (1,2,3,4,5)$ or $(2,1,3,4,5)$, and the smallest value is obtained by letting $(a,b,c,d,e) = (5,4,3,2,1)$ or $(4,5,3,2,1)$. In the former case the iterative average is $65/16$, and in the latter case the iterative average is $31/16$, so the desired difference is

$$\frac{65}{16} - \frac{31}{16} = \frac{34}{16} = \frac{17}{8}.$$

9. **Answer (A):** There were $200 \cdot 365 = 73000$ non-leap days in the 200-year time period from February 7, 1812 to February 7, 2012. One fourth of those years contained a leap day, except for 1900, so there were $\frac{1}{4} \cdot 200 - 1 = 49$ leap days during that time. Therefore Dickens was born 73049 days before a Tuesday. Because the same day of the week occurs every 7 days and $73049 = 7 \cdot 10435 + 4$, the day of Dickens' birth (February 7, 1812) was 4 days before a Tuesday, which was a Friday.

10. **Answer (D):**

The area of a triangle equals one half the product of two sides and the sine of the included angle. Because the median divides the base in half, it partitions the triangle in two triangles with equal areas. Thus $\frac{1}{2} \cdot 5 \cdot 9 \sin\theta = 15$, and $\sin\theta = \frac{2 \cdot 15}{5 \cdot 9} = \frac{2}{3}$.



OR

The altitude h to the base forms a right triangle with the median as its hypotenuse, and thus $h = 9\sin\theta$. Hence the area of the original triangle is $\frac{1}{2} \cdot 10h = \frac{1}{2} \cdot 10 \cdot 9 \sin\theta = 30$, so $\sin\theta = \frac{2 \cdot 30}{10 \cdot 9} = \frac{2}{3}$.

11. **Answer (B):**

If Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round, then the game's outcomes will be a permutation of AAAMMC, where the i^{th} letter represents the initial of the winner of the i^{th} round. There are

$$\frac{6!}{3!2!1!} = 60$$

such permutations.

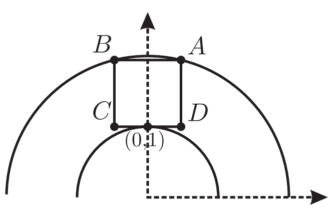
Because each round has only one winner, it follows that $P(\mathrm{M}) + P(\mathrm{C}) = 1 - P(\mathrm{A}) = \frac{1}{2}$. Also $P(\mathrm{M}) = 2P(\mathrm{C})$ and so $P(\mathrm{M}) = \frac{1}{3}$ and $P(\mathrm{C}) = \frac{1}{6}$.

The probability that Alex wins 3 rounds, Mel wins 2 rounds, and Chelsea wins 1 round is therefore

$$\frac{6!}{3!2!1!}\left(\frac{1}{2}\right)^3\left(\frac{1}{3}\right)^2\left(\frac{1}{6}\right) = \frac{60}{2^3 \cdot 3^2 \cdot 6} = \frac{5}{36}.$$

12. **Answer (D):**

Suppose by symmetry that $A = (a, b)$ with $a > 0$. Because $ABCD$ is tangent to the circle with equation $x^2 + y^2 = 1$ at $(0, 1)$ and both A and B are on the concentric circle with equation $x^2 + y^2 = 4$, it follows that $B = (-a, b)$. Then the horizontal length of the square is $2a$ and its vertical height is $b-1$. Therefore $2a = b - 1$, or $b = 2a + 1$. Substituting this into the equation $a^2 + b^2 = 4$ leads to the equation $5a^2 + 4a - 3 = 0$. By the quadratic formula, the positive root is $\frac{1}{5}(\sqrt{19} - 2)$, and so the side length $2a$ is $\frac{1}{5}(2\sqrt{19} - 4)$.



13. **Answer (D):** Let the length of the lunch break be m minutes. Then the three painters each worked $480-m$ minutes on Monday, the two helpers worked $372-m$ minutes on Tuesday, and Paula worked $672-m$ minutes on Wednesday. If Paula paints $p\%$ of the house per minute and her helpers paint a total of $h\%$ of the house per minute, then

$$\begin{aligned}(p + h)(480 - m) &= 50,\\ h(372 - m) &= 24, \text{and}\end{aligned}$$

$$p(672 - m) = 26.$$

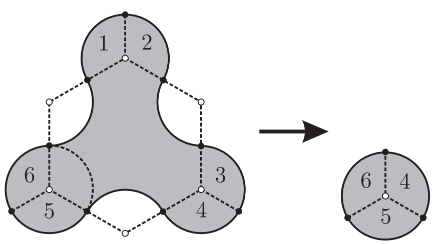
Adding the last two equations gives $672p+372h-mp-mh = 50$, and subtracting this equation from the first one gives $108h - 192p = 0$, so $h = \frac{16p}{9}$. Substitution into the first equation then leads to the system

$$\frac{25p}{9}(480 - m) = 50,$$
$$p(672 - m) = 26.$$

The solution of this system is $p = \frac{1}{24}$ and $m = 48$. Note that $h = \frac{2}{27}$.

14. **Answer (E):** The labeled circular sectors in the figure each have the same area because they are all $\frac{2\pi}{3}$-sectors of a circle of radius 1. Therefore the area enclosed by the curve is equal to the area of a circle of radius 1 plus the area of a regular hexagon of side 2. Because the regular hexagon can be partitioned into 6 congruent equilateral triangles of side 2, it follows that the required area is

$$\pi + 6\left(\frac{\sqrt{3}}{4} \cdot 2^2\right) = \pi + 6\sqrt{3}\,.$$



15. **Answer (A):** There are $2^4 = 16$ possible initial colorings for the four corner squares. If their initial coloring is $BBBB$, one of the four cyclic permutations of $BBBW$, or one of the two cyclic permutations of $BWBW$, then all four corner squares are black at the end. If the initial coloring is $WWWW$, one of the four cyclic permutations of $BWWW$, or one of the four cyclic permutations of $BBWW$, then at least one corner square is white at the end. Hence all four corner squares are black at the end with probability $\frac{7}{16}$. Similarly, all four edge squares are black at the end with probability $\frac{7}{16}$. The center square is black at the end if and only if it was initially black, so it is black at the end with probability $\frac{1}{2}$. The probability that all nine squares are black at the end is $\frac{1}{2} \cdot \left(\frac{7}{16}\right)^2 = \frac{49}{512}$.

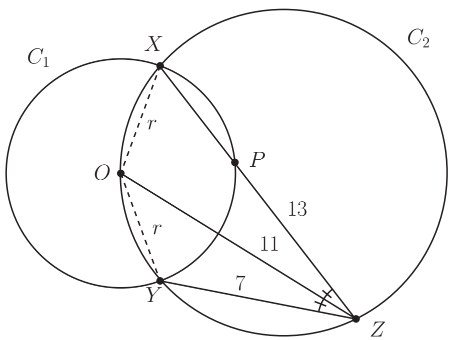
16. **Answer (E):** Let r be the radius of C_1. Because $OX = OY = r$, it follows that $\angle OZY = \angle XZO$. Applying the Law of Cosines to triangles XZO and OZY gives

$$\frac{11^2 + 13^2 - r^2}{2 \cdot 11 \cdot 13} = \cos\angle XZO = \cos\angle OZY = \frac{7^2 + 11^2 - r^2}{2 \cdot 7 \cdot 11}.$$

Solving for r^2 gives $r^2 = 30$ and so $r = \sqrt{30}$.

OR

Let P be the point on $\overline{XZ}$ such that $ZP = ZY = 7$. Because $\overline{OZ}$ is the bisector of $\angle XZY$, it follows that $\triangle OPZ \cong \triangle OYZ$. Therefore $OP = OY = r$ and thus P is on C_1. By the Power of a Point Theorem, $13 \cdot 7 = ZX \cdot ZP = OZ^2 - r^2 = 11^2 - r^2$. Solving for r^2 gives $r^2 = 30$ and so $r = \sqrt{30}$.



17. **Answer (B):** For $1 \le j \le 5$, let $S_j = \{5n + j : 0 \le n \le 5\}$. Because no pair of elements in S can have a sum that is divisible by 5, at least one of the sets $S \cap S_1$ and $S \cap S_4$ must be empty. Similarly, at least one of $S \cap S_2$ and $S \cap S_3$ must be empty, and $S \cap S_5$ can contain at most one element. Thus S can contain at most $30 - 6 - 6 - 5 = 13$ elements. An example of a set that meets the requirements is $S = \{1, 2, 6, 7, 11, 12, 16, 17, 21, 22, 26, 27, 30\}$.

OR

The set S from the previous solution shows that size 13 is possible. Consider the following partition of $\{1, 2, \ldots, 30\}$:

$$\{5,10,15,20,25,30\}, \{1,4\}, \{2,3\}, \{6,9\}, \{7,8\}, \{11,14\},$$
$$\{12,13\}, \{16,19\}, \{17,18\}, \{21,24\}, \{22,23\}, \{26,29\}, \{27,28\}.$$

There are 13 sets in this partition, and the sum of any pair of elements in the same part is a multiple of 5. Thus by the pigeon-hole principle any set S with at least 14 elements has at least two elements whose sum is divisible by 5. Therefore 13 is the largest possible size of S.

18. **Answer (A):**

Let $a = BC$, $b = AC$, and $c = AB$. Let D, E, and F be the feet of the perpendiculars from I to $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Because $\overline{BF}$ and $\overline{BD}$ are common tangent segments to the incircle of $\triangle ABC$, it follows that $BF = BD$. Similarly, $CD = CE$ and $AE = AF$. Thus

$$\begin{aligned} 2 \cdot BD = BD + BF &= (BC - CD) + (AB - AF) = BC + AB - (CE + AE) \\ &= a + c - b = 25 + 27 - 26 = 26, \end{aligned}$$

so $BD = 13$.

Let $s = \frac{1}{2}(a + b + c) = 39$ be the semiperimeter of $\triangle ABC$ and $r = DI$ the inradius of $\triangle ABC$. The area of $\triangle ABC$ is equal to rs and also equal to $\sqrt{s(s-a)(s-b)(s-c)}$ by Heron's formula. Thus

$$r^2 = \frac{(s-a)(s-b)(s-c)}{s} = \frac{14 \cdot 13 \cdot 12}{39} = 56.$$

Finally, by the Pythagorean Theorem applied to the right triangle BDI, it follows that

$$BI^2 = DI^2 + BD^2 = r^2 + BD^2 = 56 + 13^2 = 56 + 169 = 225,$$

so $BI = 15$.

19. **Answer (B):** This situation can be modeled with a graph having these six people as vertices, in which two vertices are joined by an edge if and only if the corresponding people are internet friends. Let n be the number of friends each person has; then $1 \le n \le 4$. If $n = 1$, then the graph consists of three edges sharing no endpoints. There are 5 choices for Adam's friend and then 3 ways to partition the remaining 4 people into 2 pairs of friends, for a total of $5 \cdot 3 = 15$ possibilities. The case $n = 4$ is complementary, with non-friendship playing the role of friendship, so there are 15 possibilities in that case as well.

For $n = 2$, the graph must consist of cycles, and the only two choices are two triangles (3-cycles) and a hexagon (6-cycle). In the former case, there are $\binom{5}{2} = 10$ ways to choose two friends for Adam and that choice uniquely determines the triangles. In the latter case, every permutation of the six vertices determines a hexagon, but each hexagon is counted $6 \cdot 2 = 12$ times, because the hexagon can start at any vertex and be traversed in either direction. This gives $\frac{6!}{12} = 60$ hexagons, for a total of $10 + 60 = 70$ possibilities. The complementary case $n = 3$ provides 70 more. The total is therefore $15 + 15 + 70 + 70 = 170$.

20. **Answer (B):**

A factor in the product defining $P(x)$ has degree 2012 if and only if the sum of the exponents in x is equal to 2012. Because there is only one way to write 2012 as a sum of distinct powers of 2, namely the one corresponding to its binary expansion $2012 = 11111011100_2$, it follows that the coefficient of x^{2012} is equal to $2^0 \cdot 2^1 \cdot 2^5 = 2^6$.

Note: In general, if $0 \leq n \leq 2047$ and $n = \sum_{j \in A} 2^j$ for $A \subseteq \{0, 1, 2, \ldots, 10\}$, then the coefficient of x^n is equal to 2^a where $a = \binom{11}{2} - \sum_{j \in A} j$.

21. **Answer (E):** Adding the two equations gives

$$2a^2 + 2b^2 + 2c^2 - 2ab - 2bc - 2ac = 14,$$

so

$$(a-b)^2 + (b-c)^2 + (c-a)^2 = 14.$$

Note that there is a unique way to express 14 as the sum of perfect squares (up to permutations), namely, $14 = 3^2 + 2^2 + 1^2$. Because $a - b$, $b - c$, and $c - a$ are integers with their sum equal to 0 and $a \geq b \geq c$, it follows that $a - c = 3$ and either $a - b = 2$ and $b - c = 1$, or $a - b = 1$ and $b - c = 2$. Therefore either $(a, b, c) = (c + 3, c + 1, c)$ or $(a, b, c) = (c + 3, c + 2, c)$. Substituting the relations in the first case into the first given equation yields $2011 = a^2 - c^2 + ab - b^2 = (a - c)(a + c) + (a - b)b = 3(2c + 3) + 2(c + 1)$. Solving gives $(a, b, c) = (253, 251, 250)$. The second case does not yield an integer solution. Therefore $a = 253$.

22. **Answer (C):** Label the vertices of Q as in the figure. Let m_{xy} denote the midpoint of $\overline{v_x v_y}$. Call a segment *long* if it joins midpoints of opposite edges of a face and *short* if it joins midpoints of adjacent edges.

Let p be one of the k planes. Assume p intersects the face $v_1v_2v_3v_4$. First suppose p intersects $v_1v_2v_3v_4$ by a long segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{12}m_{34}}$. Because p intersects the interior of Q, it follows that p intersects the face $v_3v_4v_8v_7$. By symmetry there are two cases: 1.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$ and 1.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

In Case 1.1 the plane p is the plane determined by the square $m_{12}m_{34}m_{78}m_{56}$. Note that p contains 4 long segments and by symmetry there are 3 planes like p, one for every pair of opposite faces of Q.

In Case 1.2 the plane p is determined by the rectangle $m_{12}m_{34}m_{48}m_{15}$. Note that p contains 2 long segments and 2 short segments, and by symmetry there are 12 planes like p, one for every edge of Q.

Second, suppose p intersects $v_1v_2v_3v_4$ by a short segment. By symmetry assume $p \cap v_1v_2v_3v_4 = \overline{m_{23}m_{34}}$. Again p must intersect the face $v_3v_4v_8v_7$. There are three cases: 2.1 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{37}}$, 2.2 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{78}}$, and 2.3 $p \cap v_3v_4v_8v_7 = \overline{m_{34}m_{48}}$.

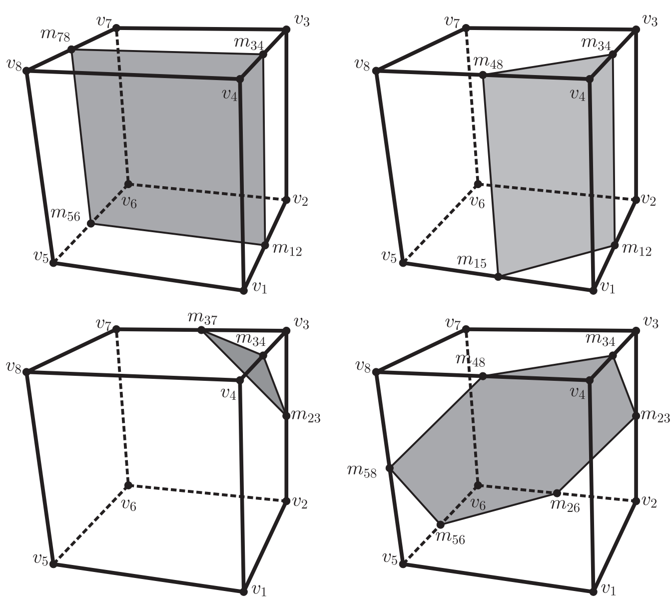
In Case 2.1 the plane p is the plane determined by the triangle $m_{23}m_{34}m_{37}$. Note that p contains 3 short segments and by symmetry there are 8 planes like p, one for every vertex of Q.

Case 2.2 duplicates Case 1.2.

In Case 2.3 the plane p is determined by the hexagon $m_{23}m_{34}m_{48}m_{58}m_{56}m_{26}$. Note that p contains 6 short segments, and by symmetry there are 4 planes like p, one for every pair of opposite vertices of Q.

Therefore the maximum possible value of k is $3 + 12 + 8 + 4 = 27$, obtained by considering all possible planes classified so far.

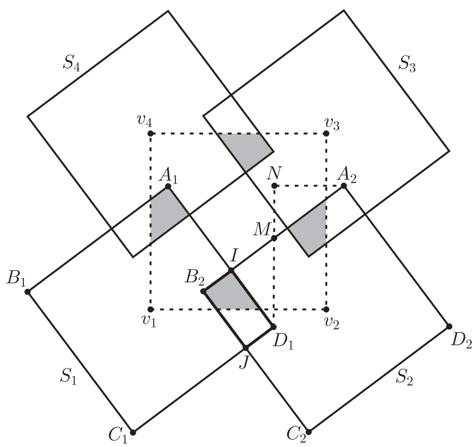
To find the minimum, note that $P \cap S$ consists of 24 short segments and 12 long segments. Every plane $p \in P$ can contain at most 6 short segments; moreover, the union of the 4 planes obtained from Case 2.3 contains all 24 short segments. Similarly, every plane $p \in P$ can contain at most 4 long segments; moreover, the union of the 3 planes obtained from Case 1.1 contains all 12 long segments. Thus the minimum possible value of k is $4 + 3 = 7$, and the required difference is $27 - 7 = 20$.



23. **Answer (C):** Consider the unit square U with vertices $v_1 = (0,0), v_2 = (1,0), v_3 = (1,1)$, and $v_4 = (0,1)$, and the squares $S_i = T(v_i)$ with $i = 1,2,3,4$. Note that $T(v)$ contains v_i if and only if $v \in S_i$. First choose a point $v = (x,y)$ uniformly at random over all pairs of real numbers (x,y) such that $0 \le x \le 1$ and $0 \le y \le 1$. In this case, the probability that $T(v)$ contains v_i and v_j is the

area of the intersection of the squares U, S_i, and S_j. This intersection is empty when v_iv_j is a diagonal of U and it is equal to Area$(U \cap S_i \cap S_j)$ when v_iv_j is a side of U. By symmetry, the probability that $T(v)$ contains two vertices of U is $4 \cdot \text{Area}(U \cap S_1 \cap S_2) = 2 \cdot \text{Area}(S_1 \cap S_2)$. By periodicity, this probability is the same as when the point $v = (x, y)$ is chosen uniformly at random over all pairs of real numbers (x, y) such that $0 \le x \le 2012$ and $0 \le y \le 2012$.

For $i = 1$ and 2, let A_i, B_i, C_i, and D_i be the vertices of S_i in counterclockwise order, where $A_1 = (0.1, 0.7)$ and $A_2 = (1.1, 0.7)$. Then $B_2 = (0.3, 0.1)$ and $D_1 = (0.7, -0.1)$. Let $M = (0.7, 0.4)$ be the midpoint of A_2B_2 and $N = (0.7, 0.7)$. Let $I \in \overline{A_2B_2}$ and $J \in \overline{C_1D_1}$ be the points of intersection of the boundaries of S_1 and S_2. Then $S_1 \cap S_2$ is the rectangle IB_2JD_1. Because D_1, M, and N are collinear and $D_1M = MA_2 = 0.5$, the right triangles A_2NM and D_1IM are congruent. Hence $ID_1 = NA_2 = 1.1 - 0.7 = 0.4$ and $IB_2 = MB_2 - MI = MB_2 - MN = 0.5 - 0.3 = 0.2$. Therefore Area$(S_1 \cap S_2) = \text{Area}(IB_2JD_1) = 0.2 \cdot 0.4 = 0.08$, and thus the required probability is 0.16.



24. **Answer (C):**

Because $y = a^x$ is decreasing for $0 < a < 1$ and $y = x^b$ is increasing on the interval $[0, \infty)$ for $b > 0$, it follows that

$$1 > a_2 = (0.2011)^{a_1} > (0.201)^{a_1} > (0.201)^1 = a_1,$$
$$a_3 = (0.20101)^{a_2} < (0.2011)^{a_2} < (0.2011)^{a_1} = a_2,$$

and

$$a_3 = (0.20101)^{a_2} > (0.201)^{a_2} > (0.201)^1 = a_1.$$

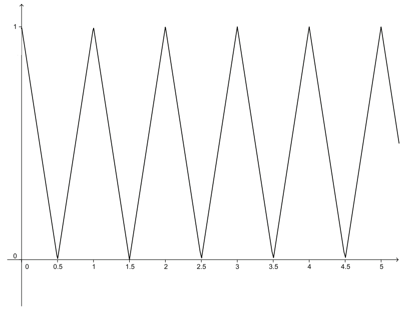
Therefore $1 > a_2 > a_3 > a_1 > 0$. More generally, it can be shown by induction that

$$1 > b_1 = a_2 > b_2 = a_4 > \cdots > b_{1005} = a_{2010}$$

$$> b_{1006} = a_{2011} > b_{1007} = a_{2009} > \cdots > b_{2011} = a_1 > 0.$$

Hence $a_k = b_k$ if and only if $2(k-1006) = 2011-k$, so $k = 1341$.

25. **Answer (C):** Because $-1 \le 2\{x\}-1 \le 1$ it follows that $0 \le f(x) \le 1$ for all $x \in \mathbb{R}$. Thus $0 \le nf(xf(x)) \le n$, and therefore all real solutions x of the required equation are in the interval $[0, n]$. Also $f(x)$ is periodic with period 1, $f(x) = 1-2x$ if $0 \le x \le \frac{1}{2}$, and $f(x) = 2x-1$ if $\frac{1}{2} \le x \le 1$. Thus the graph of $y = f(x)$ for $x \ge 0$ consists of line segments joining the points with coordinates $(k, 1)$, $(k+\frac{1}{2}, 0)$, $(k+1, 1)$ for integers $k \ge 0$ as shown.



Let a be an integer such that $0 \le a \le n-1$. Consider the interval $[a, a+\frac{1}{2})$. If $x \in [a, a+\frac{1}{2})$, then $f(x) = |2\{x\}-1| = |2(x-a)-1| = 1+2a-2x$ and thus $g(x) := xf(x) = x(1+2a-2x)$. Suppose $a \ge 1$ and $a \le x < y < a+\frac{1}{2}$. Then $2x+2y-2a-1 > 2a-1 \ge 1$ and so $(y-x)(2x+2y-2a-1) > 0$, which is equivalent to $g(x) = x(1+2a-2x) > y(1+2a-2y) = g(y)$. Thus g is strictly decreasing on $[a, a+\frac{1}{2})$ and so it maps $[a, a+\frac{1}{2})$ bijectively to $(0, a]$. Thus the graph of the function $y = f(g(x))$ on the interval $[a, a+\frac{1}{2})$ oscillates from 1 to 0 as many times as the graph of the function $y = f(x)$ on the interval $(0, a]$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a, a+\frac{1}{2})$ exactly $2a$ times.

If $a = 0$ and $x \in [a, a+\frac{1}{2})$, then $g(x) = x(1-2x)$ satisfies $0 \le g(x) \le \frac{1}{8}$, so $f(g(x)) = 1-2g(x) = 4x^2-2x+1$. If $x \in [0, \frac{1}{2})$ and $n \ge 1$, then $0 \le \frac{x}{n} < \frac{1}{2n} \le \frac{1}{2}$. Because $\frac{1}{2} \le 1-2g(x) \le 1$, it follows that the parabola $y = f(g(x))$ does not intersect any of the lines with equation $y = \frac{x}{n}$ on the interval $[0, \frac{1}{2})$.

Similarly, if $x \in [a+\frac{1}{2}, a+1)$, then $f(x) = |2\{x\}-1| = |2(x-a)-1| = 2x-2a-1$ and $g(x) := xf(x) = x(2x-2a-1)$. This time if $a+\frac{1}{2} \le x < y < a+1$, then $2x+2y-2a+1 \ge 2a+1 \ge 1$ and so $(x-y)(2x+2y-2a+1) < 0$, which is equivalent to $g(x) < g(y)$. Thus g is strictly increasing on $[a+\frac{1}{2}, a+1)$ and so it maps $[a+\frac{1}{2}, a+1)$ bijectively to $[0, a+1)$. Thus the graph of the function $y = f(g(x))$ on the interval $[a+\frac{1}{2}, a+1)$ oscillates as many times as the graph of $y = f(x)$ on the interval $[0, a+1)$. It follows that the line with equation $y = \frac{x}{n}$ intersects the graph of $y = f(g(x))$ on the interval $[a+\frac{1}{2}, a+1)$ exactly $2(a+1)$

times. Therefore the total number of intersections of the line $y = \frac{x}{n}$ and the graph of $y = f(g(x))$ is equal to

$$\sum_{a=0}^{n-1}(2a + 2(a+1)) = 2\sum_{a=0}^{n-1}(2a+1) = 2n^2.$$

Finally the smallest n such that $2n^2 \geq 2012$ is $n = 32$ because $2 \cdot 31^2 = 1922$ and $2 \cdot 32^2 = 2048$.

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