MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

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AMC 12 B

American Mathematics Contest 12 B

Wednesday, February 22, 2012

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (C):** There are $18-2=16$ more students than rabbits per classroom. Altogether there are $4\cdot 16=64$ more students than rabbits.

2. **Answer (E):** The width of the rectangle is the diameter of the circle, so the width is $2\cdot 5=10$. The length of the rectangle is $2\cdot 10=20$. Therefore the area of the rectangle is $10\cdot 20=200$.

3. **Answer (D):** Let h be the number of holes dug by the chipmunk. Then the chipmunk hid $3h$ acorns, while the squirrel hid $4(h-4)$ acorns. Since they hid the same number of acorns, $3h=4(h-4)$. Solving gives $h=16$. Thus the chipmunk hid $3\cdot 16=48$ acorns.

4. **Answer (B):** Diana's money is worth 500 dollars and Étienne's money is worth $400\cdot 1.3=520$ dollars. Hence the value of Étienne's money is greater than the value of Diana's money by

$$\frac{520-500}{500}\cdot 100\%=4\%.$$

5. **Answer (A):** The sum of two integers is even if they are both even or both odd. The sum of two integers is odd if one is even and one is odd. Only the middle two integers have an odd sum, namely $41-26=15$. Hence at least one integer must be even. A list satisfying the given conditions in which there is only one even integer is 1, 25, 1, 14, 1, 15.

6. **Answer (A):** Consider x and y as points on the real number line, with x necessarily to the right of y. Then $x-y$ is the distance between x and y. Xiaoli's rounding moved x to the right and moved y to the left. Therefore the distance between them increased, and her estimate is larger than $x-y$.

   To see that the other answer choices are not correct, let $x=2.9$ and $y=2.1$, and round each by 0.1. Then $x-y=0.8$ and Xiaoli's estimated difference is $(2.9+0.1)-(2.1-0.1)=1.0$.

7. **Answer (E):** Consider consecutive red, red, green, green, green lights as a unit. There are $5\cdot 6\cdot \frac{1}{12}=2.5$ feet between corresponding lights in successive units. The 3rd red light begins the 2nd unit, and the 21st red light begins the 11th unit. Therefore the distance between the desired lights is $(11-2)\cdot 2.5=22.5$ feet.

8. **Answer (A):** There are 3 choices for Saturday (anything except cake) and for the same reason 3 choices for Thursday. Similarly there are 3 choices for Wednesday, Tuesday, Monday, and Sunday (anything except what was to be served the following day). Therefore there are $3^6 = 729$ possible dessert menus.

OR

If any dessert could be served on Friday, there would be 4 choices for Sunday and 3 for each of the other six days. There would be a total of $4 \cdot 3^6$ dessert menus for the week, and each dessert would be served on Friday with equal frequency. Because cake is the dessert for Friday, this total is too large by a factor of 4. The actual total is $3^6 = 729$.

9. **Answer (B):** Let x be Clea's rate of walking and r be the rate of the moving escalator. Because the distance is constant, $24(x+r) = 60x$. Solving for r yields $r = \frac{3}{2}x$. Let t be the time required for Clea to make the escalator trip while just standing on it. Then $rt = 60x$, so $\frac{3}{2}xt = 60x$. Therefore $t = 40$ seconds.

10. **Answer (B):** Solve the first equation for y^2 and substitute into the second equation to get $x^2+x-20 = 0$, so $x = 4$ or $x = -5$. This leads to the intersection points $(-5, 0)$, $(4, 3)$, and $(4, -3)$. The vertical side of the triangle with these three vertices has length $3 - (-3) = 6$, and the horizontal height to that side has length $4 - (-5) = 9$, so its area is $\frac{1}{2} \cdot 6 \cdot 9 = 27$.

11. **Answer (C):** First assume $B = A - 1$. By the definition of number bases,

$$A^2 + 3A + 2 + 4(A - 1) + 3 = 6(A + A - 1) + 9.$$

Simplifying yields $A^2 - 5A - 2 = 0$, which has no integer solutions.

Next assume $B = A + 1$. In this case

$$A^2 + 3A + 2 + 4(A + 1) + 3 = 6(A + A + 1) + 9,$$

which simplifies to $A^2 - 5A - 6 = (A - 6)(A + 1) = 0$. The only positive solution is $A = 6$. Letting $A = 6$ and $B = 7$ in the original equation produces $132_6 + 43_7 = 69_{13}$, or $56 + 31 = 87$, which is true. The required sum is $A + B = 13$.

12. **Answer (E):** By symmetry, half of all such sequences end in zero. Of those, exactly one consists entirely of zeros. Each of the others contains a single subsequence of one or more consecutive ones beginning at position j and ending at position k with $1 \leq j \leq k \leq 19$. Thus the number of sequences that meet the requirements is

$$2\left(1+\sum_{k=1}^{19}\sum_{j=1}^{k}1\right) = 2(1+(1+2+3+\cdots+19)) = 2\left(1+\frac{19\cdot 20}{2}\right) = 382.$$

OR

Let A be the set of zero-one sequences of length 20 where all the zeros appear together, and let B be the equivalent set of sequences where all the ones appear together. Set A contains one sequence with no zeros and 20 sequences with exactly one zero. Each sequence of A with more than one zero has a position where the first zero appears and a position where the last zero appears, so there are $\binom{20}{2} = 190$ such sequences, and thus $|A| = 1+20+190 = 211$. By symmetry $|B| = 211$. A sequence in $A \cap B$ begins with zero and contains from 1 to 20 zeros, or it begins with one and contains from 1 to 20 ones; thus $|A \cap B| = 40$. Therefore the required number of sequences equals

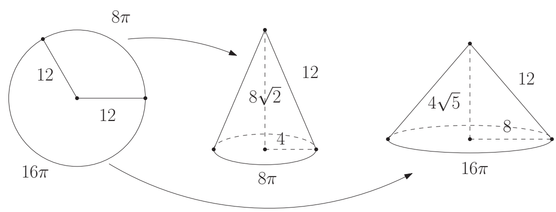
$$|A \cup B| = |A| + |B| - |A \cap B| = 211 + 211 - 40 = 382.$$

13. **Answer (D):** The parabolas have no points in common if and only if the equation $x^2+ax+b = x^2+cx+d$ has no solution. This is true if and only if the lines with equations $y = ax+b$ and $y = cx+d$ are parallel, which happens if and only if $a = c$ and $b \neq d$. The probability that $a = c$ is $\frac{1}{6}$ and the probability that $b \neq d$ is $\frac{5}{6}$, so the probability that the two parabolas have a point in common is $1 - \frac{1}{6}\cdot\frac{5}{6} = \frac{31}{36}$.

14. **Answer (A):** The smallest initial number for which Bernardo wins after one round is the smallest integer solution of $2n + 50 \geq 1000$, which is 475. The smallest initial number for which he wins after two rounds is the smallest integer solution of $2n + 50 \geq 475$, which is 213. Similarly, the smallest initial numbers for which he wins after three and four rounds are 82 and 16, respectively. There is no initial number for which Bernardo wins after more than four rounds. Thus $N = 16$, and the sum of the digits of N is 7.

15. **Answer (C):** Each sector forms a cone with slant height 12. The circumference of the base of the smaller cone is $\frac{120}{360} \cdot 2 \cdot 12 \cdot \pi = 8\pi$. Hence the radius of the base of the smaller cone is 4 and its height is $\sqrt{12^2 - 4^2} = 8\sqrt{2}$. Similarly, the circumference of the base of the larger cone is 16π. Hence the radius of the base of the larger cone is 8 and its height is $4\sqrt{5}$. The ratio of the volume of the smaller cone to the volume of larger cone is

$$\frac{\frac{1}{3}\pi \cdot 4^2 \cdot 8\sqrt{2}}{\frac{1}{3}\pi \cdot 8^2 \cdot 4\sqrt{5}} = \frac{\sqrt{10}}{10}.$$



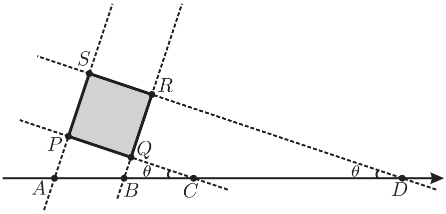
16. **Answer (B):** There are two cases to consider.

Case 1
Each song is liked by two of the girls. Then one of the three pairs of girls likes one of the six possible pairs of songs, one of the remaining pairs of girls likes one of the remaining two songs, and the last pair of girls likes the last song. This case can occur in $3 \cdot 6 \cdot 2 = 36$ ways.

Case 2
Three songs are each liked by a different pair of girls, and the fourth song is liked by at most one girl. There are $4! = 24$ ways to assign the songs to these four categories, and the last song can be liked by Amy, Beth, Jo, or no one. This case can occur in $24 \cdot 4 = 96$ ways.

The total number of possibilities is $96 + 36 = 132$.

17. **Answer (C):** Let $A = (3, 0)$, $B = (5, 0)$, $C = (7, 0)$, $D = (13, 0)$, and θ be the acute angle formed by the line PQ and the x-axis. Then $SR = PQ = AB\cos\theta = 2\cos\theta$, and $SP = QR = CD\sin\theta = 6\sin\theta$. Because $PQRS$ is a square, it follows that $2\cos\theta = 6\sin\theta$ and $\tan\theta = \frac{1}{3}$. Therefore lines SP and RQ have slope 3, and lines SR and PQ have slope $-\frac{1}{3}$. Let the points $M = (4, 0)$ and $N = (10, 0)$ be the respective midpoints of segments AB and CD. Let ℓ_1 be the line through M parallel to line SP. Let ℓ_2 be the line through N parallel to line SR. Lines ℓ_1 and ℓ_2 intersect at the center of the square $PQRS$. Line ℓ_1 satisfies the equation $y = 3(x - 4)$, and line ℓ_2 satisfies the equation $y = -\frac{1}{3}(x - 10)$. Thus the lines ℓ_1 and ℓ_2 intersect at the point $(4.6, 1.8)$, and the required sum of coordinates is 6.4 .



18. **Answer (B):** If $a_1 = 1$, then the list must be an increasing sequence. Otherwise let $k = a_1$. Then the numbers 1 through $k-1$ must appear in increasing order from right to left, and the numbers from k through 10 must appear in increasing order from left to right. For $2 \le k \le 10$ there are $\binom{9}{k-1}$ ways to choose positions in the list for the numbers from 1 through $k-1$, and the positions of the remaining numbers are then determined. The number of lists is therefore

$$1 + \sum_{k=2}^{10} \binom{9}{k-1} = \sum_{k=0}^{9} \binom{9}{k} = 2^9 = 512.$$

19. **Answer (A):** Let s be the length of the octahedron's side, and let Q_i and Q_i' be the vertices of the octahedron on $\overline{P_1P_i}$ and $\overline{P_1'P_i'}$, respectively. If Q_2 and Q_3 were opposite vertices of the octahedron, then the midpoint M of $\overline{Q_2Q_3}$ would be the center of the octahedron. Because M lies on the plane $P_1P_2P_3$, the vertex of the octahedron opposite Q_4 would be outside the cube. Therefore Q_2, Q_3, and Q_4 are all adjacent vertices of the octahedron, and by symmetry so are Q_2', Q_3', and Q_4'. For $2 \le i < j \le 4$, the Pythagorean Theorem applied to $\triangle P_1Q_iQ_j$ gives

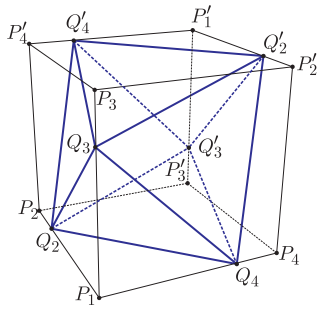
$$s^2 = (Q_iQ_j)^2 = (P_1Q_i)^2 + (P_1Q_j)^2.$$

It follows that $P_1Q_2 = P_1Q_3 = P_1Q_4 = \frac{\sqrt{2}}{2}s$, and by symmetry, $P_1'Q_2' = P_1'Q_3' = P_1'Q_4' = \frac{\sqrt{2}}{2}s$. Consequently Q_i and Q_i' are opposite vertices of the octahedron. The Pythagorean Theorem on $\triangle Q_2P_2P_3'$ and $\triangle Q_3'P_3'Q_2$ gives

$$(Q_2P_3')^2 = (P_2P_3')^2 + (Q_2P_2)^2 = 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2 \quad \text{and}$$

$$s^2 = (Q_2Q_3')^2 = (P_3'Q_3')^2 + (Q_2P_3')^2 = \left(1 - \frac{\sqrt{2}}{2}s\right)^2 + 1 + \left(1 - \frac{\sqrt{2}}{2}s\right)^2.$$

Solving for s gives $s = \frac{3\sqrt{2}}{4}$.



20. **Answer (D):** Let $ABCD$ be a trapezoid with $\overline{AB} \parallel \overline{CD}$ and $AB < CD$. Let E be the point on $\overline{CD}$ such that $CE = AB$. Then $ABCE$ is a parallelogram. Set $AB = a$, $BC = b$, $CD = c$, and $DA = d$. Then the side lengths of $\triangle ADE$ are b, d, and $c - a$. If one of b or d is equal to 11, say $b = 11$ by symmetry, then $d + (c - a) \le 7 + (5 - 3) < 11 = d$, which contradicts the triangle inequality. Thus $c = 11$. There are three cases to consider, namely, $a = 3$, $a = 5$, and $a = 7$.

If $a = 3$, then $\triangle ADE$ has side lengths 5, 7, and 8 and by Heron's formula its area is

$$\frac{1}{4}\sqrt{(5+7+8)(7+8-5)(8+5-7)(5+7-8)} = 10\sqrt{3}.$$

The area of $\triangle AEC$ is $\frac{3}{8}$ of the area of $\triangle ADE$, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{2}(35\sqrt{3})$.

If $a = 5$, then $\triangle ADE$ has side lengths 3, 6, and 7, and area

$$\frac{1}{4}\sqrt{(3+6+7)(6+7-3)(7+3-6)(3+6-7)} = 4\sqrt{5}.$$

The area of $\triangle AEC$ is $\frac{5}{6}$ of the area of $\triangle ADE$, and triangles ABC and AEC have the same area. It follows that the area of the trapezoid is $\frac{1}{3}(32\sqrt{5})$.

If $a = 7$, then $\triangle ADE$ has side lengths 3, 4, and 5. Hence this is a right trapezoid with height 3 and base lengths 7 and 11. This trapezoid has area $\frac{1}{2}(3(7+11)) = 27$.

The sum of the three possible areas is $\frac{35}{2}\sqrt{3}+\frac{32}{3}\sqrt{5}+27$. Hence $r_1 = \frac{35}{2}$, $r_2 = \frac{32}{3}$, $r_3 = 27$, $n_1 = 3$, $n_2 = 5$, and $r_1+r_2+r_3+n_1+n_2 = \frac{35}{2}+\frac{32}{3}+27+3+5 = 63+\frac{1}{6}$. Thus the required integer is 63.

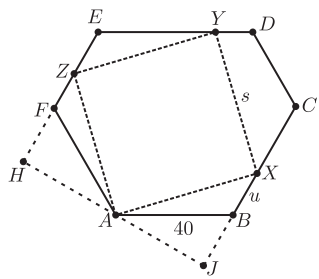
21. **Answer (A):** Extend $\overline{EF}$ to H and extend $\overline{CB}$ to J so that $\overline{HJ}$ contains A and is perpendicular to lines EF and CB. Let s be the side length of the square and let $u = BX$. Because $\angle ABJ = 60^\circ$, it follows that $BJ = 20$ and $AJ = 20\sqrt{3}$. Then by the Pythagorean Theorem

$$AX^2 = s^2 = (20+u)^2 + (20\sqrt{3})^2.$$

Because $ABCDEF$ is equiangular, it follows that $\overline{ED} \parallel \overline{AB}$ and so $\overline{EY} \parallel \overline{AB}$. Also $\overline{ZY} \parallel \overline{AX}$ and thus it follows that $\angle EYZ = \angle BAX$ and so $\triangle EYZ \cong \triangle BAX$. Thus $EZ = u$. Also, $\angle HZA = 90^\circ - \angle YZE = 90^\circ - \angle AXJ = \angle JAX$; thus $\triangle AXJ \cong \triangle ZAH$ and so $ZH = 20\sqrt{3}$ and $HA = 20 + u$. Moreover, $\angle HFA = 60^\circ$ and so $FH = \frac{HA}{\sqrt{3}} = \frac{1}{\sqrt{3}}(20 + u)$. But $EZ + ZH = EF + FH$, and so

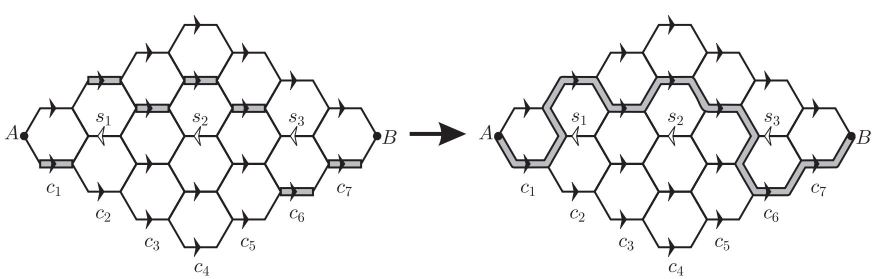
$$u + 20\sqrt{3} = 41(\sqrt{3} - 1) + \frac{20 + u}{\sqrt{3}}.$$

Solving for u yields $u = 21\sqrt{3} - 20$. Then $s^2 = (21\sqrt{3})^2 + (20\sqrt{3})^2 = 3 \cdot 29^2$ and therefore $s = 29\sqrt{3}$.

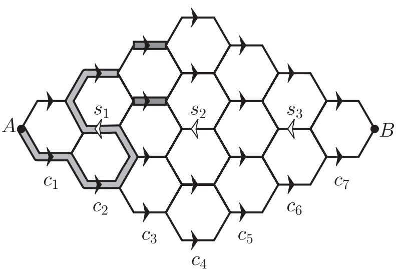


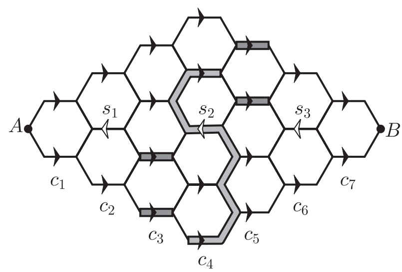
22. **Answer (E):** Label the columns having arrows as $c_1, c_2, c_3, \ldots, c_7$ according to the figure. Call those segments that can be traveled only from left to right *forward segments*. Call the segments s_1, s_2, and s_3, in columns c_2, c_4, and c_6, respectively, which can be traveled only from right to left, *back segments*. Denote S as the set of back segments traveled for a path.

First suppose that $S = \emptyset$. Because it is not possible to travel a segment more than once, it follows that the path is uniquely determined by choosing one forward segment in each of the columns c_j. There are 2, 2, 4, 4, 4, 2, and 2 choices for the forward segment in columns c_1, c_2, c_3, c_4, c_5, c_6, and c_7, respectively. This gives a total of 2^{10} total paths in this case.



Next suppose that $S = \{s_1\}$. The two forward segments in c_2, together with s_1, need to be part of the path, and once the forward segment from c_1 is chosen, the order in which the segments of c_2 are traveled is determined. Moreover, there are only 2 choices for possible segments in c_3 depending on the last segment traveled in c_2, either the bottom 2 or the top 2. For the rest of the columns, the path is determined by choosing any forward segment. Thus the total number of paths in this case is $2 \cdot 1 \cdot 2 \cdot 4 \cdot 4 \cdot 2 \cdot 2 = 2^8$, and by symmetry this is also the total for the number of paths when $S = \{s_3\}$. A similar argument gives $2 \cdot 1 \cdot 2 \cdot 4 \cdot 2 \cdot 1 \cdot 2 = 2^6$ trips for the case when $S = \{s_1, s_3\}$.





Suppose $S = \{s_2\}$. Because s_2 is traveled, it follows that 2 forward segments in c_4 need to belong to the path, one of them above s_2 (2 choices) and the other below it (2 choices). Once these are determined, there are 2 possible choices for the order in which these segments are traveled: the bottom forward segment first, then s_2, then the top forward segment, or vice versa. Next, there are only 2 possible forward segments that can be selected in c_3 and also only 2 possible forward segments that can be selected in c_5. The forward segments in c_1, c_2, c_6, and c_7 can be freely selected (2 choices each). This gives a total of $(2^3 \cdot 2 \cdot 2) \cdot 2^4 = 2^9$ paths.

If $S = \{s_1, s_2\}$, then the analysis is similar, except for the last step, where the forward segments of c_1 and c_2 are determined by the previous choices. Thus there are $(2^3 \cdot 2 \cdot 2) \cdot 2^2 = 2^7$ possibilities, and by symmetry the same number when $S = \{s_2, s_3\}$.

Finally, if $S = \{s_1, s_2, s_3\}$, then in the last step, all forward segments of c_1, c_2, c_6, and c_7 are determined by the previous choices and hence there are $2^3 \cdot 2 \cdot 2 = 2^5$ possible paths. Altogether the total number of paths is $2^{10} + 2 \cdot 2^8 + 2^6 + 2^9 + 2 \cdot 2^7 + 2^5 = 2400$.

23. **Answer (B):** If z_0^k is equal to a positive real r, then $1 = |z_0|^k = |z_0^k| = |r| = r$, so $z_0^k = 1$. Suppose that $z_0^k = 1$. If $k = 1$, then $z_0 = 1$, but $P(1) = 4 + a + b + c + d \geq 4$ so $z_0 = 1$ is not a zero of the polynomial. If $k = 2$, then $z_0 = \pm 1$. If $z_0 = -1$, then $0 = P(-1) = (4 - a) + (b - c) + d$ and by assumption $4 \geq a$, $b \geq c$, and $d \geq 0$. Thus $a = 4$, $b = c$, and $d = 0$. Conversely, if $a = 4$, $b = c$, and $d = 0$, then $P(z) = 4z^4 + 4z^3 + bz^2 + bz = z(z + 1)(4z^2 + b)$ satisfies the required conditions. If $k = 3$, then $z_0 = 1$ or $z_0 = \gamma$ where γ is any of the roots of $\gamma^2 + \gamma + 1 = 0$. If $z_0 = \gamma$, then $0 = P(\gamma) = 4\gamma + a + b(-1 - \gamma) + c\gamma + d =$

$(a-b)+d+\gamma((4-b)+c)$ and by assumption $a \ge b$, $d \ge 0$, $4 \ge b$, and $c \ge 0$. Thus $a = b$, $d = 0$, $b = 4$, and $c = 0$. Conversely, if $a = b = 4$ and $c = d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 = 4z^2(z^2 + z + 1)$ satisfies the given conditions because $z_0 = \cos(2\pi/3) + i\sin(2\pi/3)$ is a zero of this polynomial. If $k = 4$, then $z_0 = \pm 1$ or $z_0 = \pm i$. If $z_0 = \pm i$, then $0 = P(\pm i) = 4 \mp ia - b \pm ic + d = (4-b) + d \mp i(a-c)$ and by assumption $4 \ge b$, $d \ge 0$, and $4 \ge a \ge b \ge c$. Thus $b = 4$, $d = 0$, and $a = c = 4$. Conversely, if $a = b = c = 4$ and $d = 0$, then $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z = 4z(z+1)(z^2+1)$ satisfies the given conditions, but it was already considered in the case when $z_0 = -1$. The remaining case is that z_0^k is not a positive real number for $1 \le k \le 4$. In this case,

$$4z^5 - (z-1)P(z) = z^4(4-a) + z^3(a-b) + z^2(b-c) + z(c-d) + d.$$

If $z = z_0$, then the triangle inequality yields

$$\begin{aligned} 4 &= \left|z_0^4(4-a) + z_0^3(a-b) + z_0^2(b-c) + z_0(c-d) + d\right| \\ &\le \left|z_0^4(4-a)\right| + \left|z_0^3(a-b)\right| + \left|z_0^2(b-c)\right| + \left|z_0(c-d)\right| + |d| \\ &= |z_0|^4(4-a) + |z_0|^3(a-b) + |z_0|^2(b-c) + |z_0|(c-d) + d \\ &= 4 - a + a - b + b - c + c - d + d = 4. \end{aligned}$$

Thus equality must occur throughout. This means that the vectors $v_4 = z_0^4(4-a)$, $v_3 = z_0^3(a-b)$, $v_2 = z_0^2(b-c)$, $v_1 = z_0(c-d)$, and $v_0 = d$ are parallel and they belong to the same quadrant. If two of these vectors are nonzero, then the quotient must be a positive real number; but dividing the vector with the largest exponent of z_0 by the other would yield a positive rational number times z_0^k for some $1 \le k \le 4$. Because not all of the v_j can be zero, it follows that there is exactly one of them that is nonzero. If $v_0 = d \ne 0$ and $v_1 = v_2 = v_3 = v_4 = 0$, then $4 = a = b = c = d$, and $P(z) = 4z^4 + 4z^3 + 4z^2 + 4z + 4$ satisfies the given conditions because $z_0 = \cos(2\pi/5) + i\sin(2\pi/5)$ is a zero of this polynomial. Finally, if $v_j \ne 0$ for some $1 \le j \le 4$ and the rest are zero, then $4z_0^5 = v_j = z_0^j n$ for some positive integer n, and so $z_0^{5-j} = \frac{1}{4}n$ is a positive real.

Therefore the complete list of polynomials is: $4z^4 + 4z^3 + 4z^2 + 4z + 4$, $4z^4 + 4z^3 + 4z^2$, and $4z^4 + 4z^3 + bz^2 + bz$ with $0 \le b \le 4$. The required sum is $20 + 12 + \sum_{b=0}^{4}(8 + 2b) = 32 + 40 + (2 + 4 + 6 + 8) = 92$.

24. **Answer (D):** Let $S_N = (f_1(N), f_2(N), f_3(N), \ldots)$. If N_1 divides N_2, then $f_1(N_1)$ divides $f_1(N_2)$. Thus S_{N_2} is unbounded if S_{N_1} is unbounded. Call N *essential* if S_N is unbounded and $N \le 400$ is not divisible by any smaller number n such that S_n is unbounded. Assume $N = p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is essential. If $e_j = 1$ for some j, then $f_1(N) = f_1(\frac{N}{p_j})$. Let $n = \frac{N}{p_j}$ and note that S_N and S_n coincide after the first term and consequently S_n is unbounded. This contradicts the fact that N is essential. Thus $e_j \ge 2$ for all $1 \le j \le k$. Moreover, $(p_1p_2\cdots p_k)^2 \le p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k} = N \le 400$; thus $p_1p_2\cdots p_k \le \sqrt{400} = 20$. Because $2 \cdot 3 \cdot 5 > 20$ it follows that $k \le 2$.

First analyze the case when $n = 2^a \cdot 3^b$. In that case $f_2(n) = f_1(2^{2b-2} \cdot 3^{a-1}) = 2^{2a-4} \cdot 3^{2b-3}$; thus S_n is unbounded if and only if $a \ge 5$ or $b \ge 4$, and n is essential if and only if $n = 2^5$ or $n = 3^4$.

If $k = 1$, then $N = p^e$ for some prime $p \le 19$. The cases $p = 2$ or $p = 3$ have been considered before. If $p = 5$, then $f_1(5^a) = 2^{a-1} \cdot 3^{a-1}$ and because $a \le 3$, no power of 5 in the given range is essential. If $p = 7$, then $f_1(7^a) = 2^{3a-3}$, and thus $N = 7^3$ is essential. If $p \ge 11$, then $p^3 > 400$. Because $f_1(11^2) = 2^2 \cdot 3$, $f_2(13^2) = f_1(2 \cdot 7) = 1$, $f_1(17^2) = 2 \cdot 3^2$, and $f_2(19^2) = f_1(2^2 \cdot 5) = 3$, no powers of 11, 13, 17, or 19 are essential.

If $k = 2$, then the only possible pairs of primes (p_1, p_2) are $(2, 3)$, $(2, 5)$, $(2, 7)$, and $(3, 5)$. The pair $(2, 3)$ was analyzed before and it yields no essential N. If $N = 2^a \cdot 5^b \le 400$ is essential, then $2 \le a \le 4$ and $b = 2$. Moreover $f_1(N) = 2 \cdot 3^a$, so $a = 4$ and thus only $N = 2^4 \cdot 5^2$ is essential in this case. If $(p_1, p_2) = (2, 7)$ or $(3, 5)$ and $N = p_1^{e_1} p_2^{e_2} \le 400$ is essential, then $N \in \{2^2 \cdot 7^2, 2^3 \cdot 7^2, 3^2 \cdot 5^2\}$. Because $f_1(2^2 \cdot 7^2) = 2^3 \cdot 3$, $f_1(2^3 \cdot 7^2) = 2^3 \cdot 3^2$, and $f_1(3^2 \cdot 5^2) = 2^3 \cdot 3$, it follows that there are no essential N in this case.

Therefore the only essential values of N are $2^5 = 32$, $3^4 = 81$, $7^3 = 343$, and $2^4 \cdot 5^2 = 400$. These values have $\lfloor \frac{400}{32} \rfloor = 12$, $\lfloor \frac{400}{81} \rfloor = 4$, $\lfloor \frac{400}{343} \rfloor = 1$, and $\lfloor \frac{400}{400} \rfloor = 1$ multiples, respectively, in the range $1 \le N \le 400$. Because there are no common multiples, the required answer is $12 + 4 + 1 + 1 = 18$.

25. **Answer (B):** First note that the isosceles right triangles t can be excluded from the product because $f(t) = 1$ for these triangles. All triangles mentioned from now on are scalene right triangles. Let $O = (0, 0)$. First consider all triangles $t = \triangle ABC$ with vertices in $S \cup \{O\}$. Let R_1 be the reflection with respect to the line with equation $x = 2$. Let $A_1 = R_1(A)$, $B_1 = R_1(B)$, $C_1 = R_1(C)$, and $t_1 = \triangle A_1B_1C_1$. Note that $\triangle ABC \cong \triangle A_1B_1C_1$ with right angles at A and A_1, but the counterclockwise order of the vertices of t_1 is A_1, C_1, and B_1. Thus $f(t_1) = \tan(\angle A_1C_1B_1) = \tan(\angle ACB)$ and

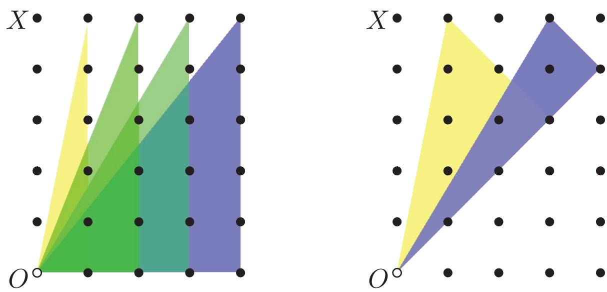
$$f(t)f(t_1) = \tan(\angle CBA)\tan(\angle ACB) = \frac{AC}{AB} \cdot \frac{AB}{AC} = 1.$$

The reflection R_1 is a bijection of $S \cup \{O\}$ and it induces a partition of the triangles in pairs (t, t_1) such that $f(t)f(t_1) = 1$. Thus the product over all triangles in $S \cup \{O\}$ is equal to 1, and thus the required product is equal to the reciprocal of $\prod_{t \in T_1} f(t)$, where T_1 is the set of triangles with vertices in $S \cup \{O\}$ having O as one vertex.

Let $S_1 = \{(x, y) : x \in \{0, 1, 2, 3, 4\}$, and $y \in \{0, 1, 2, 3, 4\}\}$ and let R_2 be the reflection with respect to the line with equation $x = y$. For every right triangle $t = \triangle OBC$ with vertices B and C in S_1, let $B_2 = R_2(B)$, $C_2 = R_2(C)$, and $t_2 = \triangle OB_2C_2$. Similarly as before, R_2 is a bijection of S_1 and it induces a partition of the triangles in pairs (t, t_2) such that $f(t)f(t_2) = 1$. Thus

$\prod_{t\in T_1} f(t) = \prod_{t\in T_2} f(t)$, where T_2 is the set of triangles with vertices in $S\cup\{O\}$ with O as one vertex, and another vertex with y coordinate equal to 5.

Next, consider the reflection R_3 with respect to the line with equation $y = \frac{5}{2}$. Let $X = (0,5)$. For every right triangle $t = \triangle OXC$ with C in S, let $C_3 = R_3(C)$, and $t_3 = \triangle OXC_3$. As before R_3 induces a partition of these triangles in pairs (t, t_3) such that $f(t)f(t_3) = 1$. Therefore to calculate $\prod_{t\in T_2} f(t)$, the only triangles left to consider are the triangles of the form $t = \triangle OYZ$ where $Y \in \{(x,5) : x \in \{1,2,3,4\}\}$ and $Z \in S \setminus \{X\}$.



The following argument shows that there are six such triangles. Because the y coordinate of Y is greater than zero, the right angle of t is not at O. The slope of the line OY has the form $\frac{5}{x}$ with $1 \le x \le 4$, so if the right angle were at Y, then the vertex Z would need to be at least 5 horizontal units away from Y, which is impossible. Therefore the right angle is at Z. There are 4 such triangles with Z on the x-axis, with vertices $O, Z = (x,0)$, and $Y = (x,5)$ for $1 \le x \le 4$. There are two more triangles: with vertices O, $Z = (3,3)$, and $Y = (1,5)$, and with vertices O, $Z = (4,4)$, and $Y = (3,5)$. The product of the values $f(t)$ over these six triangles is equal to

$$\frac{1}{5}\cdot\frac{2}{5}\cdot\frac{3}{5}\cdot\frac{4}{5}\cdot\frac{3\sqrt{2}}{2\sqrt{2}}\cdot\frac{4\sqrt{2}}{\sqrt{2}} = \frac{144}{625}.$$

Thus the required product equals

$$\prod_{t\in T} f(t) = \left(\prod_{t\in T_1} f(t)\right)^{-1} = \left(\prod_{t\in T_2} f(t)\right)^{-1} = \left(\frac{144}{625}\right)^{-1} = \frac{625}{144}.$$

The problems and solutions in this contest were proposed by Bernardo Abrego, Betsy Bennett, Barb Currier, Steve Davis, Zuming Feng, Silvia Fernández, Peter Gilchrist, Jerrold Grossman, Dan Kennedy, Joe Kennedy, David Wells, LeRoy Wenstrom.

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