MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

64th Annual

AMC 12 B

American Mathematics Contest 12 B

Wednesday, February 20, 2013

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

Prof. Bernardo M. Abrego

1. **Answer (C):** The difference between the high and low temperatures was 16 degrees, so the difference between each of these and the average temperature was 8 degrees. The low temperature was 8 degrees less than the average, so it was $3^\circ - 8^\circ = -5^\circ$.

2. **Answer (A):** The garden is $2 \cdot 15 = 30$ feet wide and $2 \cdot 20 = 40$ feet long. Hence Mr. Green expects $\frac{1}{2} \cdot 30 \cdot 40 = 600$ pounds of potatoes.

3. **Answer (D):** The number 201 is the 1^{st} number counted when proceeding backwards from 201 to 3. In turn, 200 is the 2^{nd} number, 199 is the 3^{rd} number, and x is the $(202-x)^{\text{th}}$ number. Therefore 53 is the $(202-53)^{\text{th}}$ number, which is the 149^{th} number.

4. **Answer (B):** Let D equal the distance traveled by each car. Then Ray's car uses $\frac{D}{40}$ gallons of gasoline and Tom's car uses $\frac{D}{10}$ gallons of gasoline. The cars combined miles per gallon of gasoline is

$$\frac{2D}{\left(\frac{D}{40} + \frac{D}{10}\right)} = 16.$$

5. **Answer (C):** The sum of all the ages is $55 \cdot 33 + 33 \cdot 11 = 33 \cdot 66$, so the average of all the ages is

$$\frac{33 \cdot 66}{55 + 33} = \frac{33 \cdot 66}{88} = \frac{33 \cdot 3}{4} = 24.75\,.$$

6. **Answer (B):** By completing the square the equation can be rewritten as follows:

$$x^2 + y^2 = 10x - 6y - 34,$$
$$x^2 - 10x + 25 + y^2 + 6y + 9 = 0,$$
$$(x-5)^2 + (y+3)^2 = 0.$$

Therefore $x = 5$ and $y = -3$, so $x + y = 2$.

7. **Answer (E):** Note that Jo starts by saying 1 number, and this is followed by Blair saying 2 numbers, then Jo saying 3 numbers, and so on. After someone completes her turn after saying the number n, then $1+2+3+\cdots+n = \frac{1}{2}n(n+1)$

numbers have been said. If $n = 9$ then 45 numbers have been said. Therefore there are $53 - 45 = 8$ more numbers that need to be said. The 53^{rd} number said is 8.

8. **Answer (B):** The solution to the system of equations $3x - 2y = 1$ and $y = 1$ is $B = (x, y) = (1, 1)$. The perpendicular distance from A to $\overline{BC}$ is 3. The area of $\triangle ABC$ is $\frac{1}{2} \cdot 3 \cdot BC = 3$, so $BC = 2$. Thus point C is 2 units to the right or to the left of $B = (1, 1)$. If $C = (-1, 1)$ then the line AC is vertical and the slope is undefined. If $C = (3, 1)$, then the line AC has slope $\frac{1-(-2)}{3-(-1)} = \frac{3}{4}$.

9. **Answer (C):** Because $12! = 2^{10} \cdot 3^5 \cdot 5^2 \cdot 7 \cdot 11$, the largest perfect square that divides $12!$ is $2^{10} \cdot 3^4 \cdot 5^2$ which has square root $2^5 \cdot 3^2 \cdot 5$. The sum of the exponents is $5 + 2 + 1 = 8$.

10. **Answer (E):** After Alex makes m exchanges at the first booth and n exchanges at the second booth, Alex has $75-(2m-n)$ red tokens, $75-(3n-m)$ blue tokens, and $m+n$ silver tokens. No more exchanges are possible when he has fewer than 2 red tokens and fewer than 3 blue tokens. Therefore no more exchanges are possible if and only if $2m - n \geq 74$ and $3n - m \geq 73$. Equality can be achieved when $(m, n) = (59, 44)$, and Alex will have $59 + 44 = 103$ silver tokens.

    Note that the following exchanges produce 103 silver tokens:

	Red Tokens	Blue Tokens	Silver Tokens
Exchange 75 blue tokens	100	0	25
Exchange 100 red tokens	0	50	75
Exchange 48 blue tokens	16	2	91
Exchange 16 red tokens	0	10	99
Exchange 9 blue tokens	3	1	102
Exchange 2 red tokens	1	2	103

11. **Answer (A):** Suppose that the two bees start at the origin and that the positive directions of the x, y, and z coordinate axes correspond to the directions east, north, and up, respectively. Note that the bees are always getting farther apart from each other. After bee A has traveled 7 feet it will have gone 3 feet north, 2 feet east, and 2 feet up. Its position would be the point $(2, 3, 2)$. In the same time bee B will have gone 4 feet south and 3 feet west, and its position would be the point $(-3, -4, 0)$. This puts them at a distance

$$\sqrt{(2-(-3))^2 + (3-(-4))^2 + 2^2} = \sqrt{78} < 10$$

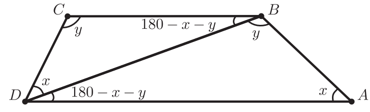
After this moment, bee A will travel east to the point $(3, 3, 2)$ and bee B will travel west to the point $(-4, -4, 0)$. Their distance after traveling one foot will be

$$\sqrt{(3-(-4))^2 + (3-(-4))^2 + 2^2} = \sqrt{102} > 10.$$

Hence bee A is traveling east and bee B is traveling west when they are exactly 10 feet away from each other.

12. **Answer (D):** Cities C and E and the roads leading in and out of them can be replaced by a second A–D road and a second B–D road, respectively. If routes are designated by the list of cities they visit in order, then there are 4 types of routes: $ABDADB$, $ADABDB$, $ADBADB$, and $ADBDAB$. Each type of route represents 4 actual routes, because the trip between A and D can include the detour through E either the first or the second time, and a similar choice applies for the trip between B and D. Therefore there are $4 \cdot 4 = 16$ different routes.

13. **Answer (D):** Let the degree measures of the angles be as shown in the figure. The angles of a triangle form an arithmetic progression if and only if the median angle is $60°$, so one of x, y, or $180 - x - y$ must be equal to 60. By symmetry of the role of the triangles ABD and DCB, assume that $x \le y$. Because $x \le y < 180 - x$ and $x < 180 - y \le 180 - x$, it follows that the arithmetic progression of the angles in $ABCD$ from smallest to largest must be either $x, y, 180-y, 180-x$ or $x, 180-y, y, 180-x$. Thus either $x+180-y = 2y$, in which case $3y = x + 180$; or $x + y = 2(180 - y)$, in which case $3y = 360 - x$. Neither of these is compatible with $y = 60$ (the former forces $x = 0$ and the latter forces $x = 180$), so either $x = 60$ or $x + y = 120$.



First suppose that $x = 60$. If $3y = x + 180$, then $y = 80$, and the sequence of angles in $ABCD$ is $(x, y, 180 - y, 180 - x) = (60, 80, 100, 120)$. If $3y = 360 - x$, then $y = 100$, and the sequence of angles in $ABCD$ is $(x, 180 - y, y, 180 - x) = (60, 80, 100, 120)$. Finally, suppose that $x+y = 120$. If $3y = x+180$, then $y = 75$, and the sequence of angles in $ABCD$ is $(x, y, 180-y, 180-x) = (45, 75, 105, 135)$. If $3y = 360 - x$, then $y = 120$ and $x = 0$, which is impossible.

Therefore, the sum in degrees of the two largest possible angles is $105 + 135 = 240$.

14. **Answer (C):** Let the two sequences be (a_n) and (b_n), and assume without loss of generality that $a_1 < b_1$. The definitions of the sequences imply that

$a_7 = 5a_1 + 8a_2 = 5b_1 + 8b_2$, so $5(b_1 - a_1) = 8(a_2 - b_2)$. Because 5 and 8 are relatively prime, 8 divides $b_1 - a_1$ and 5 divides $a_2 - b_2$. It follows that $a_1 \le b_1 - 8 \le b_2 - 8 \le a_2 - 13$. The minimum value of N results from choosing $a_1 = 0$, $b_1 = b_2 = 8$, and $a_2 = 13$, in which case $N = 104$.

15. **Answer (B):** The prime factorization of 2013 is $3 \cdot 11 \cdot 61$. There must be a factor of 61 in the numerator, so $a_1 \ge 61$. Since $a_1!$ will have a factor of 59 and 2013 does not, there must be a factor of 59 in the denominator, and $b_1 \ge 59$. Thus $a_1 + b_1 \ge 120$, and this minimum value can be achieved only if $a_1 = 61$ and $b_1 = 59$. Furthermore, this minimum value is attainable because

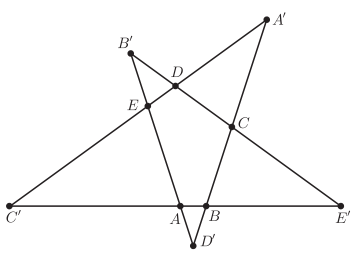
$$2013 = \frac{(61!)(11!)(3!)}{(59!)(10!)(5!)}.$$

Thus $|a_1 - b_1| = a_1 - b_1 = 61 - 59 = 2$.

16. **Answer (A):** The sum of the internal angles of the pentagon $ABCDE$ is $3 \cdot 180^\circ = 540^\circ$ and by assumption all internal angles are equal, so they are all equal to $\frac{1}{5}(540^\circ) = 108^\circ$. Therefore the supplementary angles at each of the vertices are all equal to $180^\circ - 108^\circ = 72^\circ$. It follows that all the triangles making up the points of the star are isosceles triangles with angles measuring 72°, 72°, and 36°. Label the rest of the vertices of the star as in the figure. By the above argument, there is a constant c such that $A'C = A'D = c \cdot CD$ and similar expressions for the other four points of the star. Therefore the required perimeter equals

$$A'C + A'D + B'D + B'E + C'A + C'E + D'A + D'B + E'B + E'C = \\ 2c(CD + DE + EA + AB + BC) = 2c,$$

and therefore the maximum and minimum values are the same and their difference is 0.



Note: The constant c equals $\frac{1}{2}\csc(\frac{\pi}{10}) = \frac{1}{2}(\sqrt{5} + 1)$.

17. **Answer (D):**

From the equations, $a + b = 2 - c$ and $a^2 + b^2 = 12 - c^2$. Let x be an arbitrary real number, then $(x-a)^2 + (x-b)^2 \geq 0$; that is, $2x^2 - 2(a+b)x + (a^2+b^2) \geq 0$. Thus

$$2x^2 - 2(2-c)x + (12 - c^2) \geq 0$$

for all real values x. That means the discriminant $4(2-c)^2 - 4 \cdot 2(12-c^2) \leq 0$. Simplifying and factoring gives $(3c-10)(c+2) \leq 0$. So the range of values of c is $-2 \leq c \leq \frac{10}{3}$. Both maximum and minimum are attainable by letting $(a, b, c) = (2, 2, -2)$ and $(a, b, c) = (-\frac{2}{3}, -\frac{2}{3}, \frac{10}{3})$. Therefore the difference between the maximum and minimum possible values of c is $\frac{10}{3} - (-2) = \frac{16}{3}$.

18. **Answer (B):**

If the game starts with 2013 coins and Jenna starts, then she picks 3 coins, and then no matter how many Barbara chooses, Jenna responds by keeping the number of remaining coins congruent to 0 (mod 5). That is, she picks 3 if Barbara picks 2, and she picks 1 if Barbara picks 4. This ensures that on her last turn Jenna will leave 0 coins and thus she will win. Similarly, if Barbara starts, then Jenna can reply as before so that the number of remaining coins is congruent to 3 (mod 5). On her last turn Barbara will have 3 coins available. She is forced to remove 2 and thus Jenna will win by taking the last coin.

If the game starts with 2014 coins and Jenna starts, then she picks 1 coin and reduces the game to the previous case of 2013 coins where she wins. If Barbara starts, she selects 4 coins. Then no matter what Jenna chooses, Barbara responds by keeping the number of remaining coins congruent to 0 (mod 5). This ensures that on her last turn Barbara will leave 0 coins and win the game. Thus whoever goes first will win the game with 2014 coins.

19. **Answer (B):** The Pythagorean Theorem applied to right triangles ABD and ACD gives $AB^2 - BD^2 = AD^2 = AC^2 - CD^2$; that is, $13^2 - BD^2 = 15^2 - (14 - BD)^2$, from which it follows that $BD = 5$, $CD = 9$, and $AD = 12$. Because triangles AED and ADC are similar,

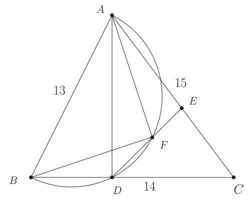
$$\frac{AE}{12} = \frac{DE}{9} = \frac{12}{15},$$

implying that $ED = \frac{36}{5}$ and $AE = \frac{48}{5}$.

Because $\angle AFB = \angle ADB = 90^\circ$, it follows that $ABDF$ is cyclic. Thus $\angle ABD + \angle AFD = 180^\circ$ from which $\angle ABD = \angle AFE$. Therefore right triangles ABD and AFE are similar. Hence

$$\frac{FE}{5} = \frac{\frac{48}{5}}{12},$$

from which it follows that $FE = 4$. Consequently $DF = DE - FE = \frac{36}{5} - 4 = \frac{16}{5}$.



20. **Answer (A):** Because $135^\circ < x < 180^\circ$, it follows that $\cos x < 0 < \sin x$ and $|\sin x| < |\cos x|$. Thus $\tan x < 0$, $\cot x < 0$, and

$$|\tan x| = \frac{|\sin x|}{|\cos x|} < 1 < \frac{|\cos x|}{|\sin x|} = |\cot x|\,.$$

Therefore $\cot x < \tan x$. Moreover, $\cot x = \frac{\cos x}{\sin x} < \cos x$. Thus the four vertices P, Q, R, and S are located on the parabola $y = x^2$ and P and S are in between Q and R. If $\overline{AB}$ and $\overline{CD}$ are chords on the parabola $y = x^2$ such that the x-coordinates of A and B are less than the x-coordinates of C and D, then the slope of $\overline{AB}$ is less than the slope of $\overline{CD}$. It follows that the two parallel sides of the trapezoid must be $\overline{QR}$ and $\overline{PS}$. Thus the slope of $\overline{QR}$ is equal to the slope of $\overline{PS}$. Thus,

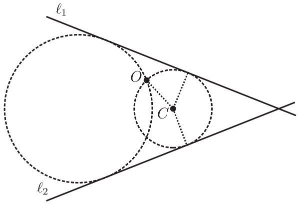
$$\cot x + \sin x = \tan x + \cos x.$$

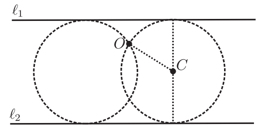
Multiplying by $\sin x \cos x \neq 0$ and rearranging gives the equivalent identity

$$(\cos x - \sin x)\,(\cos x + \sin x - \sin x \cos x) = 0.$$

Because $\cos x - \sin x \neq 0$ in the required range, it follows that $\cos x + \sin x - \sin x \cos x = 0$. Squaring and using the fact that $2 \sin x \cos x = \sin(2x)$ gives $1 + \sin(2x) = \frac{1}{4}\sin^2(2x)$. Solving this quadratic equation in the variable $\sin(2x)$ gives $\sin(2x) = 2 \pm 2\sqrt{2}$. Because $-1 < \sin 2x < 1$, the only solution is $\sin(2x) = 2 - 2\sqrt{2}$. There is indeed such a trapezoid for $x = 180^\circ + \frac{1}{2}\arcsin(2 - 2\sqrt{2}) \approx 152.031^\circ$.

21. **Answer (C):** If the directrices of two parabolas with the same focus intersect, then the corresponding parabolas intersect in exactly two points. The same conclusion holds if the directrices are parallel and the focus is between the two lines. Moreover, if the directrices are parallel and the focus is not between the two lines, then the corresponding parabolas do not intersect. Indeed, a point C belongs to the intersection of the parabolas with focus O and directrices ℓ_1 and ℓ_2, if and only if, $d(C, \ell_1) = OC = d(C, \ell_2)$. That is, the circle with center C and radius OC is tangent to both ℓ_1 and ℓ_2. If ℓ_1 and ℓ_2 are parallel and O is not between them, then clearly such circle does not exist. If ℓ_1 and ℓ_2 intersect and O is not on them, then there are are exactly two circles tangent to both ℓ_1 and ℓ_2 that go through O. The same is true if ℓ_1 and ℓ_2 are parallel and O is between them.

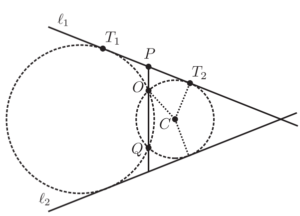




Thus there are $\binom{30}{2}$ pairs of parabolas and the pairs that do not intersect are exactly those whose directrices have the same slope and whose y-intercepts have the same sign. There are 5 different slopes and $2 \cdot \binom{3}{2} = 6$ pairs of y-intercepts with the same sign taken from $\{-3, -2, -1, 1, 2, 3\}$. Because the pairs of parabolas that intersect do so at exactly two points and no point is in three parabolas, it follows that the total number of intersection points is

$$2\left(\binom{30}{2} - 5 \cdot 6\right) = 810.$$

Note: It is possible to construct the two circles through O and tangent to the lines ℓ_1 and ℓ_2 as follows: Let ℓ' be the bisector of the angle determined by the angular sector spanned by ℓ_1 and ℓ_2 that contains O (or the midline of ℓ_1 and ℓ_2 if these lines are parallel and O is between them). Let Q be the symmetric point of O with respect to ℓ' and let P be the intersection of ℓ_1 and the line OQ (if $O = Q$ then let P be the intersection of ℓ_1 and a perpendicular line to ℓ' by O). If C is one of the desired circles, then C passes through O and Q and is tangent to ℓ_1. Let T be the point of tangency of C and ℓ_1. By the Power of a Point Theorem, $PT^2 = PO \cdot PQ$. The circle with center P and radius $\sqrt{PO \cdot PQ}$ intersects ℓ_1 in two points T_1 and T_2. The circumcircles of OQT_1 and OQT_2 are the desired circles.



22. **Answer (A):**

Using the change of base identity gives $\log n \cdot \log_n x = \log x$ and $\log m \cdot \log_m x = \log x$. The equivalent equation is

$$(\log x)^2 - \frac{1}{8}(7\log m + 6\log n)\log x - \frac{2013}{8}\log m \cdot \log n = 0.$$

As a quadratic equation in $\log x$, the sum of the two solutions $\log x_1$ and $\log x_2$ is equal to the negative of the linear coefficient. It follows that

$$\log(x_1x_2) = \log x_1 + \log x_2 = \frac{1}{8}(7\log m + 6\log n) = \log\left(\left(m^7n^6\right)^{1/8}\right).$$

Let $N = x_1x_2$ be the product of the solutions. Suppose p is a prime dividing m. Let p^a and p^b be the largest powers of p that divide m and n respectively. Then p^{7a+6b} is the largest power of p that divides $m^7n^6 = N^8$. It follows that $7a+6b \equiv 0 \pmod 8$. If a is odd, then there is no solution to $7a+6b \equiv 0 \pmod 8$ because $7a$ is not divisible by $\gcd(6,8) = 2$. If $a \equiv 0 \pmod 8$, then because $a > 0$, it follows that $N^8 = m^7n^6 \ge (p^8)^7 = p^{56} \ge 2^{56}$, so $N \ge 2^7 = 128$. If $a \equiv 2 \pmod 8$ then $14 + 6b \equiv 0 \pmod 8$ is equivalent to $3b + 3 \equiv 3b + 7 \equiv 0 \pmod 4$. Thus $b \equiv 3 \pmod 4$ and then $N^8 = m^7n^6 \ge (p^2)^7(p^3)^6 = p^{32} \ge 2^{32}$, so $N \ge 2^4 = 16$ with equality for $m = 2^2$ and $n = 2^3$. Finally, if $a \ge 4$ and a is not a multiple of 8, then $b \ge 1$ and thus $N^8 = m^7n^6 \ge (p^4)^7(p^1)^6 = p^{34} \ge 2^{34}$, so $N \ge 2^{17/4} > 2^4 = 16$. Therefore the minimum product is $N = 16$ obtained uniquely when $m = 2^2$ and $n = 2^3$. The requested sum is $m + n = 4 + 8 = 12$.

23. **Answer (E):** Expand the set of three-digit positive integers to include integers N, $0 \le N \le 99$, with leading zeros appended. Because $\operatorname{lcm}(5^2, 6^2, 10^2) = 900$, such an integer N meets the required condition if and only if $N + 900$ does. Therefore N can be considered to be chosen from the set of integers between 000 and 899, inclusive. Suppose that the last two digits in order of the base-5 representation of N are a_1 and a_0. Similarly, suppose that the last two digits of the base-6 representation of N are b_1 and b_0. By assumption, $2N \equiv a_0 + b_0 \pmod{10}$, but $N \equiv a_0 \pmod 5$ and so

$$a_0 + b_0 \equiv 2N \equiv 2a_0 \pmod{10}.$$

Thus $a_0 \equiv b_0 \pmod{10}$ and because $0 \le a_0 \le 4$ and $0 \le b_0 \le 5$, it follows that $a_0 = b_0$. Because $N \equiv a_0 \pmod 5$, it follows that there is an integer N_1 such that $N = 5N_1 + a_0$. Also, $N \equiv a_0 \pmod 6$ implies that $5N_1 + a_0 \equiv a_0 \pmod 6$ and so $N_1 \equiv 0 \pmod 6$. It follows that $N_1 = 6N_2$ for some integer N_2 and so $N = 30N_2 + a_0$. Similarly, $N \equiv 5a_1 + a_0 \pmod{25}$ implies that $30N_2 + a_0 \equiv 5a_1 + a_0 \pmod{25}$ and then $N_2 \equiv 6N_2 \equiv a_1 \pmod 5$. It follows that $N_2 = 5N_3 + a_1$ for some integer N_3 and so $N = 150N_3 + 30a_1 + a_0$. Once more, $N \equiv 6b_1 + a_0 \pmod{36}$ implies that $6N_3 - 6a_1 + a_0 \equiv 150N_3 + 30a_1 + a_0 \equiv 6b_1 + a_0 \pmod{36}$ and then $N_3 \equiv a_1 + b_1 \pmod 6$. It follows that $N_3 = 6N_4 + a_1 + b_1$ for some integer N_4 and so $N = 900N_4 + 180a_1 + 150b_1 + a_0$. Finally, $2N \equiv 10(a_1 + b_1) + 2a_0 \pmod{100}$ implies that

$$60a_1 + 2a_0 \equiv 360a_1 + 300b_1 + 2a_0 \equiv 10a_1 + 10b_1 + 2a_0 \pmod{100}.$$

Therefore $5a_1 \equiv b_1 \pmod{10}$, equivalently, $b_1 \equiv 0 \pmod 5$ and $a_1 \equiv b_1 \pmod 2$. Conversely, if $N = 900N_4 + 180a_1 + 150b_1 + a_0$, $a_0 = b_0$, and $5a_1 \equiv b_1 \pmod{10}$, then $2N \equiv 60a_1 + 2a_0 = 10(a_1 + 5a_1) + a_0 + b_0 \equiv 10(a_1 + b_1) + (a_0 + b_0) \pmod{100}$. Because $0 \le a_1 \le 4$ and $0 \le b_1 \le 5$, it follows that there are exactly 5 different pairs (a_1, b_1), namely $(0, 0)$, $(2, 0)$, $(4, 0)$, $(1, 5)$, and $(3, 5)$. Each of these can be combined with 5 different values of a_0 $(0 \le a_0 \le 4)$, to determine exactly 25 different numbers N with the required property.

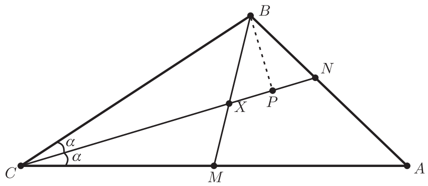
24. **Answer (A):** Let $\alpha = \angle ACN = \angle NCB$ and $x = BN$. Because $\triangle BXN$ is equilateral it follows that $\angle BXC = \angle CNA = 120°$, $\angle CBX = \angle BAC = 60° - \alpha$, and $\angle CBA = \angle BMC = 120° - \alpha$. Thus $\triangle ABC \sim \triangle BMC$ and $\triangle ANC \sim \triangle BXC$. Then

$$\frac{BC}{2} = \frac{BC}{AC} = \frac{MC}{BC} = \frac{1}{BC},$$

so $BC = \sqrt{2}$; and

$$\frac{CX + x}{2} = \frac{CN}{AC} = \frac{CX}{BC} = \frac{CX}{\sqrt{2}},$$

so $CX = (\sqrt{2} + 1)x$.



Let P be the midpoint of $\overline{XN}$. Because $\triangle BXN$ is equilateral, the triangle BPC is a right triangle with $\angle BPC = 90°$. Then by the Pythagorean Theorem,

$$\begin{aligned} 2 = BC^2 = CP^2 + PB^2 &= (CX + XP)^2 + PB^2 \\ &= \left(CX + \frac{1}{2}BN\right)^2 + \left(\frac{\sqrt{3}}{2}BN\right)^2 \\ &= \left(\sqrt{2} + \frac{3}{2}\right)^2 x^2 + \left(\frac{\sqrt{3}}{2}\right)^2 x^2 = (5 + 3\sqrt{2})x^2. \end{aligned}$$

Therefore

$$x^2 = \frac{2}{5 + 3\sqrt{2}} = \frac{10 - 6\sqrt{2}}{7}.$$

OR

Establish as in the first solution that $CX = (\sqrt{2}+1)x$. Then the Law of Cosines applied to $\triangle BCX$ gives

$$2 = BC^2 = BX^2 + CX^2 - 2BX \cdot CX \cdot \cos(120°)$$

$$= x^2 + (1+\sqrt{2})^2x^2 + (1+\sqrt{2})x^2$$
$$= (5+3\sqrt{2})x^2,$$

and solving for x^2 gives the requested answer.

25. **Answer (B):** Let $P(z)$ be a polynomial in G. Because the coefficients of $P(z)$ are real, it follows that the nonreal roots of $P(z)$ must be paired by conjugates; that is, if $a+ib$ is a root, then $a-ib$ is a root as well. In particular, $P(z)$ can be factored into the product of pairwise different linear polynomials of the form $(z-c)$ with $c \in \mathbb{Z}$ and quadratic polynomials of the form $(z-(a+ib))(z-(a-ib)) = z^2-2az+(a^2+b^2)$ with $a, b \in \mathbb{Z}$ and $b \neq 0$. Moreover, the product of the independent terms of these polynomials must be equal to 50, so each of a^2+b^2 or c must be a factor of 50. Call these linear or quadratic polynomials *basic* and for every $d \in \{1, 2, 5, 10, 25, 50\}$, let B_d be the set of basic polynomials with independent term equal to $\pm d$.

The equation $a^2+b^2 = 1$ has a pair of conjugate solutions in integers with $b \neq 0$, namely $(a, b) = (0, \pm 1)$. Thus there is only 1 basic quadratic polynomial with independent term of magnitude 1: $(z-i)(z+i) = z^2+1$. Similarly, the equation $a^2+b^2 = 2$ has 2 pairs of conjugate solutions with $b \neq 0$, $(a, b) = (1, \pm 1)$ and $(-1, \pm 1)$. These give the following 2 basic polynomials with independent term ± 2: $(z-1-i)(z-1+i) = z^2-2z+2$ and $(z+1+i)(z+1-i) = z^2+2z-2$. In the same way the equations $a^2+b^2 = 5$, $a^2+b^2 = 10$, $a^2+b^2 = 25$, and $a^2+b^2 = 50$ have 4, 4, 5, and 6 respective pairs of conjugate solutions (a, b). These are $(2, \pm 1)$, $(-2, \pm 1)$, $(1, \pm 2)$, and $(-1, \pm 2)$; $(3, \pm 1)$, $(-3, \pm 1)$, $(1, \pm 3)$, and $(-1, \pm 3)$; $(3, \pm 4)$, $(-3, \pm 4)$, $(4, \pm 3)$, $(-4, \pm 3)$, and $(0, \pm 5)$; and $(7, \pm 1)$, $(-7, \pm 1)$, $(1, \pm 7)$, $(-1, \pm 7)$, $(5, \pm 5)$, and $(-5, \pm 5)$. These generate all possible basic quadratic polynomials with nonreal roots and independent term that divides 50. The basic linear polynomials with real roots are $z-c$ where $c \in \{\pm 1, \pm 2, \pm 5, \pm 10, \pm 25, \pm 50\}$. Thus the linear basic polynomials contribute 2 to $|B_d|$. It follows that $|B_1| = 3$, $|B_2| = 4$, $|B_5| = 6$, $|B_{10}| = 6$, $|B_{25}| = 7$, and $|B_{50}| = 8$.

Because P has independent term 50, there are either 8 choices for a polynomial in B_{50}, or $7 \cdot 4$ choices for a product of two polynomials, one in B_{25} and the other in B_2, or $6 \cdot 6$ choices for a product of two polynomials, one in B_{10} and the other in B_5, or $4 \cdot \binom{6}{2}$ choices for a product of three polynomials, one in B_2 and the other two in B_5. Finally, each of the polynomials $z+1$ and z^2+1 in B_1 can appear or not in the product, but the presence of the polynomial $z-1$ is determined by the rest: if the product of the remaining independent terms is -50, then it has to be present, and if the product is 50, then it must not be in the product. Thus, the grand total is

$$2^2\left(8+7\cdot 4+6\cdot 6+4\cdot\binom{6}{2}\right) = 2^2(8+28+36+60) = 4\cdot 132 = 528.$$

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